DIFFERENTIAL EQUATIONS CONTAINING ARBITRARY FUNCTIONS*

BY

GILBERT AMES BLISS

A recent study of the problem of computing differential corrections for a trajectory has led the writer to the present investigation of the properties of solutions of differential equations considered as functions of other arbitrary functions which enter into the differential equations themselves. In ballistics it is customary to compute first the coördinates of a trajectory, uninfluenced by wind or other disturbances, as solutions of the differential equations of motion. A new system of differential equations is then set up which accounts for the disturbances of the trajectory, and with the help of which corrections to the coördinates of the projectile on the original trajectory can be computed. The new equations involve arbitrary functions and the corrections found are themselves functions of these functions. The details of this situation will be described in another paper. In the following pages the results attained are of a more general character than is necessary for the ballistic problem, and it seems probable that they will be useful in many other connections.

The differential equations here considered have the form

$$\frac{dx^{(i)}}{d\tau} = f^{(i)}(\tau, x^{(1)}, \cdots, x^{(p)}) \qquad (i = 1, 2, \cdots, p), \tag{1}$$

and it is supposed that a particular solution

$$x^{(i)} = u^{(i)}(\tau) \qquad (i = 1, \cdots, p;\ \tau_1 \leqq \tau \leqq \tau_2) \tag{2}$$

for a particular set of functions $f^{(i)}$ is known. When the functions $f^{(i)}$ themselves are allowed to vary the solutions of these equations are functions $x^{(i)}[\tau, \tau_0, x_0, f]$ of the variable τ, of the coördinates τ_0 and

$$x_0 = (x_0^{(1)}, \cdots, x_0^{(p)})$$

of a prescribed initial point (τ, x_0) of the curve, and of the functions $f^{(i)}$ in the second members of the equations. In Sections 1 and 2 below the character of these solutions in a neighborhood of the particular solution (2) are considered.

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It is proved that they are continuous and have differentials with respect to all their arguments, and the parts of the differentials which involve the variations of the functions f are differentials in a sense analogous to that of Fréchet or Volterra in the theory of functions of lines.* The variations of the functions f considered in this paper are arbitrary ones. In special cases, however, they may be caused by the presence of variable parameters in the second members of the equations (1), or by the presence of arbitrary functions as is the case in ballistics.

In Section 3 some implicit function theorems are developed which have an application in the theory of the wind and density corrections in ballistics. The character of a solution $\tau = \tau[\tau_0, x_0, f]$ of an equation of the form

$$\phi[\tau, \tau_0, x_0, f] = 0$$

is discussed. Under suitable hypotheses the solution τ will be continuous and have a differential, as will also the function obtained by substituting the solution τ in a second function ψ of the same sort as ϕ. The theorems deduced are special cases of some much more general ones in general analysis.

The notations and properties of multi-partite numbers and matrices have been used freely throughout. This gives a somewhat forbidding aspect to the pages for one who has not considered the advantages of these notations. It is a matter of a short time only, however, to familiarize one's self with their essential properties, and the instruments so provided are of very great usefulness and power in many applications. The use of multi-partite numbers and matrices in the differential equation theory was inaugurated by Peano† and the present writer has more recently listed in some detail‡ the properties which are found to be most useful in this connection. The notations and methods of the paper last referred to, and which will hereafter be designated as the paper (A), will be preserved as far as possible. For example, one-partite numbers will always be represented by Greek letters, p-partite numbers by small Roman letters, and matrices by capital Roman letters. A list of the notations used is given on page 21 of the paper (A). It will assist in the interpretation of the following equations if it is noted that the properties of the moduli of multi-partite numbers and matrices are analogous to those of the ordinary absolute value.

* Fréchet, *Sur la notion de différentielle d'une fonction de ligne*, these Transactions, vol. 15 (1914), p. 139. Volterra, *Leçons sur les fonctions des lignes*, p. 25.

† Mathematische Annalen, vol. 32 (1888), p. 450.

‡ *The solutions of differential equations of the first order as functions of their initial values*, Annals of Mathematics, 2d series, vol. 6 (1905), p. 58; *Solutions of differential equations as functions of the constants of integration*, Bulletin of the American Mathematical Society, 2d series, vol. 24 (1918), p. 21.

1. Solutions of differential equations containing arbitrary functions

The system (1) of differential equations is represented in matrix notation by the single equation

$$\frac{dx}{d\tau} = f(\tau, x). \tag{3}$$

Interpreted from the standpoint of multi-partite numbers the latter means that a similar equation holds for each value of the index $i = 1, 2, \cdots, p$ of equations (1). An initial solution

$$x = u(\tau) \qquad (\tau_1 \leqq \tau \leqq \tau_2) \tag{C}$$

of the differential equations (3) for a special set f' of functions f, is supposed to be known in advance, each function of the set u having a continuous derivative on the interval $\tau_1 \tau_2$. There is also a neighborhood C_ϵ of the arc C, consisting of all the points (τ, x) satisfying the conditions

$$\tau_1 \leqq \tau \leqq \tau_2, \qquad \mod [x - u(\tau)] < \epsilon, \tag{C_ϵ}$$

in which all the functions f to be considered are single valued and continuous and satisfy the Lipschitz conditions

$$\mod [f(\tau, x') - f(\tau, x)] < \kappa \mod (x' - x) \tag{4}$$

with a prescribed constant κ which is the same for all the f's considered. This would, for example, be the case for all functions f having suitably bounded continuous first derivatives in C_ϵ.

Under the circumstances described in the preceding paragraph a sequence of sets of approximation functions $v_m = (v_m^{(1)}, \cdots, v_m^{(p)})$ may be defined by the equations

$$\begin{aligned} v_0[\tau, \tau_0, x_0, f] &= u(\tau) + x_0 - u(\tau_0), \\ v_{m+1}[\tau, \tau_0, x_0, f] &= x_0 + \int_{\tau_0}^{\tau} f(\tau, v_m)\, d\tau, \end{aligned} \tag{5}$$

and each set of functions v_m has the initial values x_0 when $\tau = \tau_0$.

To justify the use of these formulas it must first be shown, as in the paper (A), that there is a domain R of elements $[\tau, \tau_0, x_0, f]$ in which every one of the sets v_m is well defined and provides only points (τ, v_m) in the region C_ϵ where the continuity properties of the functions f have been presupposed. Such a domain R is defined by the conditions

$$\tau_1 \leqq \tau \leqq \tau_2, \qquad (\tau_0, x_0) \text{ in } C_{\delta/2}, \qquad \mod (f - f') \leqq \frac{\delta}{2\lambda}, \tag{R}$$

where λ is the length of the interval $\tau_1 \tau_2$ and $\delta = \epsilon e^{-\kappa\lambda}$. For the functions u clearly satisfy the equations

$$u(\tau) = u(\tau_0) + \int_{\tau_0}^{\tau} f'(\tau, u)\, d\tau,$$

and from equations (5), (4), and properties of the modulus analogous to those of the absolute value, it follows that in the region R

$$\text{mod}\,(v_0 - u) = \text{mod}\,[x_0 - u(\tau_0)] \leqq \frac{\delta}{2} < \delta,$$

$$\begin{aligned}
\text{mod}\,(v_1 - u) &\leqq \text{mod}\,[x_0 - u(\tau_0)] + \text{mod} \int_{\tau_0}^{\tau} [f(\tau, v_0) - f'(\tau, u)]\, d\tau \\
&\leqq \frac{\delta}{2} + \left| \int_{\tau_0}^{\tau} \text{mod}\,[f(\tau, v_0) - f'(\tau, v_0)]\, d\tau \right| \\
&\quad + \left| \int_{\tau_0}^{\tau} \text{mod}\,[f'(\tau, v_0) - f'(\tau, u)]\, d\tau \right| \\
&\leqq \delta + \left| \int_{\tau_0}^{\tau} \kappa\, \text{mod}\,(v_0 - u)\, d\tau \right| \\
&\leqq \delta \left\{ 1 + \frac{\kappa|\tau - \tau_0|}{1!} \right\}.
\end{aligned}$$

A simple inductive proof now gives

$$\text{mod}\,(v_m - u) \leqq \delta \left\{ 1 + \frac{\kappa|\tau - \tau_0|}{1!} + \cdots + \frac{\kappa^m|\tau - \tau_0|^m}{m!} \right\} \leqq \delta e^{\kappa\lambda} = \epsilon,$$

which shows that the points (τ, v_m) are in C_ϵ for every m and every set of arguments $[\tau, \tau_0, x_0, f]$ in R.

The proofs (1) that the sequence of functionals $v_m[\tau, \tau_0, x_0, f]$ converges uniformly in the domain R to a limiting set $v[\tau, \tau_0, x_0, f]$, (2) that this limiting set satisfies the equations

$$v[\tau, \tau_0, x_0, f] = x_0 + \int_{\tau_0}^{\tau} f(\tau, v)\, d\tau$$

and hence also the original equations (3), and (3) that when τ_0, x_0, f are prescribed the functions v are the only ones which satisfy equations (3), are now precisely those of the paper (A) cited above, and they need not be reproduced here.

The approximation functionals $v_m[\tau, \tau_0, x_0, f]$ have uniform continuity in the domain R of the following type, designated in the rest of this paper as *continuity of type I*. Let ϕ be one of them, and designate by $\Delta\phi$ the difference

$$\Delta\phi = \phi[\tau + \Delta\tau, \tau_0 + \Delta\tau_0, x_0 + \Delta x_0, f + \Delta f] - \phi[\tau, \tau_0, x_0, f].$$

Then for every positive constant ϵ' there is a second constant δ' such that

$|\Delta\phi| < \epsilon'$ whenever the sets $[\tau + \Delta\tau,\ \tau_0 + \Delta\tau_0,\ x_0 + \Delta x_0,\ f + \Delta f]$ and $[\tau, \tau_0, x_0, f]$ are in R and

$$|\Delta\tau|,\ |\Delta\tau_0|,\ \text{mod}\,\Delta x_0,\ \text{mod}\,\Delta f < \delta'.$$

The last inequality is supposed to hold uniformly for all points (τ, x) in C_ϵ.

The proof of this property of uniform continuity is an inductive one. The function v_0 is evidently continuous in the sense just described, and it is assumed for purposes of proof that v_m has the same property. It follows then from equations (5) that

$$\begin{aligned}\Delta v_{m+1} &= \Delta x_0 + \int_{\tau_0+\Delta\tau_0}^{\tau+\Delta\tau} \bar{f}(\tau, v_m + \Delta v_m)\,d\tau - \int_{\tau_0}^{\tau} f(\tau, v_m)\,\Delta\tau \\ &= \Delta x_0 + \int_{\tau}^{\tau+\Delta\tau} \bar{f}(\tau, v_m + \Delta v_m)\,d\tau + \int_{\tau_0+\Delta\tau_0}^{\tau_0} \bar{f}(\tau, v_m + \Delta v_m)\,d\tau \\ &\quad + \int_{\tau_0}^{\tau} \Delta f(\tau, v_m + \Delta v_m)\,d\tau + \int_{\tau_0}^{\tau} [f(\tau, v_m + \Delta v_m) - f(\tau, v_m)]\,d\tau,\end{aligned}$$

where $\bar{f} = f + \Delta f$. Hence

$$\begin{aligned}\text{mod}\,\Delta v_{m+1} \leqq\ &\text{mod}\,\Delta x_0 + \mu[|\Delta\tau| + |\Delta\tau_0|] \\ &+ \left|\int_{\tau_0}^{\tau} \text{mod}\,\Delta f d\tau\right| + \left|\int_{\tau_0}^{\tau} \kappa\ \text{mod}\,\Delta v_m d\tau\right|,\end{aligned}$$

where μ is a maximum of mod f in the domain R, and the sum of these terms can clearly be made less than ϵ' by sufficiently restricting δ'. It follows at once from the uniform convergence of the sequence $\{v_m\}$ that its limit function $v[\tau, \tau_0, x_0, f]$ has also continuity of type I in the domain R.

Suppose now that the class of sets of functions f is still further restricted to include only functions which have continuous first derivatives in the neighborhood C_ϵ, and that these derivatives have a common prescribed upper bound which is the same for all the f's considered. Then there will exist a constant κ such that the Lipschitz condition (4) holds for all of the sets of functions. The proofs of the paper (A) repeatedly cited establish for each f the existence of the derivatives of the solutions given by the equations

$$\frac{\partial v}{\partial x_0} = I + \int_{\tau_0}^{\tau} f_x\,d\tau + \cdots + \int_{\tau_0}^{\tau} f_x \cdots \int_{\tau_0}^{\tau} f_x\,d\tau^m + \cdots, \tag{6}$$

$$\frac{\partial v}{\partial \tau_0} = -\frac{\partial v}{\partial x_0} f(\tau_0, x_0). \tag{7}$$

In the former of these equations $\partial v/\partial x_0$ is the matrix $||\partial v^{(i)}/\partial x_0^{(j)}||$ $(i, j = 1, \cdots, p)$ of derivatives of the elements of v with respect to the elements of x_0,

I is the identity matrix, f_x is the matrix of derivatives of the elements of f with respect to those of x with the arguments $(\tau, x) = (\tau, v)$, and the equation holds for every pair of corresponding elements of the matrices on the two sides when the sums and products and integrals on the right are interpreted in the usual way. In the latter equation $\partial v/\partial \tau_0$ is the multi-partite number whose elements are the derivatives of the elements of v with respect to the scalar τ_0, and the second member is the negative product of the matrix $\partial v/\partial x_0$ by the multi-partite number $f(\tau_0, x_0)$.

The sum (6) converges uniformly in R, as follows at once from the proof given in the paper (A) cited above. In order therefore to show that the derivatives $\partial v/\partial x_0$ are uniformly continuous in R it is only necessary to show that each term is uniformly continuous. The continuity to be demonstrated is, however, of a special kind which will be called *continuity of type II*. Let $\phi[\tau, \tau_0, x_0, f]$ be one of the derivatives. It is to be proved uniformly continuous in the sense that for every positive constant ϵ' there is a second positive constant δ' depending only upon f and ϵ' and such that

$$|\Delta\phi| = |\phi[\tau + \Delta\tau, \tau_0 + \Delta\tau_0, x_0 + \Delta x_0, f + \Delta f] - \phi[\tau, \tau_0, x_0, f]| < \epsilon$$

whenever the two sets $[\tau + \Delta\tau, \tau_0 + \Delta\tau_0, x_0 + \Delta x_0, f + \Delta f]$ and $[\tau, \tau_0, x_0, f]$ are in R and the inequalities

$$|\Delta\tau|, \ |\Delta\tau_0|, \ \mathrm{mod}\ \Delta x_0, \ \mathrm{mod}\ \Delta f, \ \mathrm{mod}\ \Delta f_x < \delta', \tag{8}$$

are true. The last two are understood as holding uniformly for all arguments (τ, x) of the elements of Δf and Δf_x in C_x. In order that $\Delta\phi$ shall approach zero, therefore, it is necessary according to this definition, that the *derivatives* of the elements of Δf with respect to the elements of x shall approach zero as well as the elements of $\Delta\tau, \Delta\tau_0, \Delta x_0, \Delta f$. Furthermore the uniformity of the continuity is with respect to the arguments τ, τ_0, x_0 but not with respect to f, as is indicated by the fact that the constant δ' depends upon f as well as upon ϵ'.

It is clear that $f_x(\tau, v)$ is continuous in the sense just described. For let $\bar{f} = f + \Delta f$, and let Δv stand for the difference

$$\Delta v = v[\tau + \Delta\tau, \tau_0 + \Delta\tau_0, x_0 + \Delta x_0, f + \Delta f] - v[\tau, \tau_0, x_0, f].$$

Then

$$\begin{aligned}
\mathrm{mod}\,[\bar{f}_x(\tau + \Delta\tau, v + \Delta v) - f_x(\tau, v)] &\leqq \mathrm{mod}\,[\bar{f}_x(\tau + \Delta\tau, v + \Delta v) - f_x(\tau + \Delta\tau, v + \Delta v)] \\
&\quad + \mathrm{mod}\,[f_x(\tau + \Delta\tau, v + \Delta v) - f_x(\tau, v)] \\
&\leqq \mathrm{mod}\ \Delta f_x(\tau + \Delta\tau, v + \Delta v) \\
&\quad + \mathrm{mod}\,[f_x(\tau + \Delta\tau, v + \Delta v) - f_x(\tau, v)].
\end{aligned}$$

The next to last term on the right approaches zero with mod Δf_x, and the last term does the same with the first four of the quantities (8), since the elements of f_x and v are continuous. But the uniformity of the approach to zero in the case of the last modulus is with respect to τ, τ_0, x_0 only, since a δ' suitable for making that modulus less than ϵ' will in general depend upon f. It is easily provable that

$$\int_{\tau_0}^{\tau} f_x \, d\tau, \qquad f_x \int_{\tau_0}^{\tau} f_x \, d\tau$$

are continuous in the same way, and so by induction that each term of the series (6) has this property. Equations (7) with the equations

$$\frac{\partial v}{\partial \tau} = f(\tau, v)$$

now show that all the elements of the first derivatives of both v and $\partial v/\partial \tau$ have continuity of type II. These results may be summarized in the following theorem:

THEOREM. *Let the arc*

$$(C) \qquad x = u(\tau) \qquad (\tau_1 \leqq \tau \leqq \tau_2)$$

be a solution of the differential equations

$$\frac{dx}{d\tau} = f'(\tau, x),$$

where the functions $f'(\tau, x)$ are continuous and satisfy the Lipschitz conditions

$$(9) \qquad \operatorname{mod}[f'(\tau, x') - f'(\tau, x)] < \kappa \operatorname{mod}(x' - x)$$

in a neighborhood C_ϵ of the arc C. Then for every other set of functions f having these properties in C_ϵ with the same constant κ there exists a unique set of solutions

$$(10) \qquad x = v[\tau, \tau_0, x_0, f]$$

through each initial point (τ_0, x_0) sufficiently near to C. The functionals $v[\tau, \tau_0, x_0, f]$ so defined are single valued and have uniform continuity of the type I described above in the domain R of sets $[\tau, \tau_0, x_0, f]$ defined by the conditions

$$(R) \quad \tau_1 \leqq \tau \leqq \tau_2, \qquad (\tau_0, x_0) \text{ in } C_{\delta/2}, \qquad \operatorname{mod}(f' - f) \leqq \frac{\delta}{2(\tau_2 - \tau_1)}.$$

If the class of functions f considered is restricted to include only those with continuous first derivatives whose absolute values have a common upper bound in the neighborhood C_ϵ, then the functionals (10) and their derivatives $\partial v/\partial \tau$ have partial derivatives with respect to the elements of τ, τ_0, x_0 which possess continuity of type II in the domain R.

The results which have been proved above can be extended immediately to the case where the functions f contain a set of parameters $a = (a_1, \cdots, a_q)$. The class of functions $f(\tau, x, a)$ now to be considered is the totality of those which are continuous and satisfy the Lipschitz condition (9) with a fixed κ for sets of arguments (τ, x, a) such that (τ, x) is in C_ϵ and a in a region $\mathfrak{A}$ of points $(a_1, \cdots, a_q)$. The proofs already given show that the approximations sets $v_m[\tau, \tau_0, x_0, f, a]$ are uniformly continuous and converge uniformly in a region R_1 of sets $[\tau, \tau_0, x_0, f, a]$ satisfying the conditions

$$(R_1) \qquad [\tau, \tau_0, x_0, f] \text{ is in } R, \ a \text{ in } \mathfrak{A},$$

and the limit function $v[\tau, \tau_0, x_0, f, a]$ has uniform continuity of type I in R_1. The conclusions of the theorem hold true in each case when R is replaced by R_1.

Consider as a special case the class of functions f which are linear in x of the form $f = Ax$, where A is a matrix whose elements are functions of τ and parameters $a = (a_1, \cdots, a_q)$. Let the elements of A be restricted to be continuous and to be such that $\operatorname{mod} A < \kappa$ for all arguments (τ, a) with $\tau_1 \leqq \tau \leqq \tau_2$ and a in a region $\mathfrak{A}$. Then the Lipschitz condition

$$\operatorname{mod}[f(\tau, x') - f(\tau, x)] = \operatorname{mod} A(x' - x)$$
$$\leqq \operatorname{mod} A \operatorname{mod}(x' - x) \leqq \kappa \operatorname{mod}(x' - x)$$

is satisfied for all these functions and they are all continuous in every neighborhood C_ϵ of the special arc

$$(C) \qquad x = 0 \qquad (\tau_1 \leqq \tau \leqq \tau_2)$$

satisfying the equations (3) with the functions $f'(\tau, x) \equiv 0$ substituted. For the system (3) with such sets of functions $f = Ax$ the proofs of the preceding paragraphs establish the existence and continuity of the solutions $v[\tau, \tau_0, x_0, A, a]$ in a region R_2 of elements $[\tau, \tau_0, x_0, A, a]$ satisfying the conditions

$$(R_2) \qquad \tau_1 \leqq \tau \leqq \tau_2, \qquad \tau_1 \leqq \tau_0 \leqq \tau_2, \qquad \operatorname{mod} x_0 \leqq \delta,$$

A arbitrary except for the restrictions just specified, a in $\mathfrak{A}$.

The constant δ is in this case entirely arbitrary, for it was originally restricted in order to make the points (τ, v_m) lie in the neighborhood C_ϵ where the properties of the functions f were presupposed, and when the elements of f are linear in x this neighborhood is C_∞. The proof that

$$\operatorname{mod}(v_m - u) = \operatorname{mod} v_m \leqq \delta e^{\kappa\lambda}$$

is, however, still valid, so that the points (τ, v_m) all lie in the neighborhood C_ϵ for which $\epsilon = \delta e^{\kappa\lambda}$. The continuity of $x[\tau, \tau_0, x_0, A, a]$ in this case is such that for every ϵ' there exists a δ' such that

$$\text{mod } (x[\tau+\Delta\tau, \tau_0+\Delta\tau_0, x_0+\Delta x_0, A+\Delta A, a+\Delta a] - x[\tau, \tau_0, x_0, A, a]) < \epsilon'$$

whenever the two sets of arguments of x are in R_2 and

$$|\Delta\tau|,\ |\Delta\tau_0|,\ \text{mod } \Delta x_0,\ \text{mod } \Delta A,\ \text{mod } \Delta a < \delta'.$$

2. Difference functions and differentials of the solutions

Under the assumptions made in the last paragraph of the theorem of the preceding section the functionals $x[\tau, \tau_0, x_0, f]$ have not only continuous derivatives with respect to the arguments τ, τ_0, x_0 but also differentials with respect to the arguments f in a sense analogous to that of the theory of functions of lines. As a first step toward the proof of this statement it is to be shown that *for every pair of sets of elements* $[\tau, \tau_0, x_0, f]$, $[\tau, \tau_0, x_0, \bar{f}]$ *of the region* R *there exists a matrix of functions* $L[\tau', \tau, \tau_0, x_0, f, \bar{f}]$ *such that the differences*

$$\Delta v = v[\tau, \tau_0, x_0, \bar{f}] - v[\tau, \tau_0, x_0, f]$$

are expressible in the form

$$\Delta v = \int_{\tau_0}^{\tau} L[\tau', \tau, \tau_0, x_0, f, \bar{f}]\, \Delta f(\tau', v[\tau', \tau_0, x_0, f])\, d\tau', \tag{11}$$

where $\Delta f = \bar{f} - f$. The elements of L have continuity of type II for all arguments $[\tau', \tau, \tau_0, x_0, f, \bar{f}]$ satisfying the conditions

$$\tau_1 \leqq \tau' \leqq \tau_2, \qquad [\tau, \tau_0, x_0, f] \text{ and } [\tau, \tau_0, x_0, \bar{f}] \text{ in } R. \tag{12}$$

An application of Taylor's theorem with integral form of remainder term* shows that the differences Δv satisfy the differential equations

$$\begin{aligned} \frac{d}{d\tau}\Delta v &= \bar{f}(\tau, v+\Delta v) - f(\tau, v) \\ &= \bar{f}(\tau, v+\Delta v) - \bar{f}(\tau, v) + \Delta f(\tau, v) \\ &= A\Delta v + \Delta f(\tau, v), \end{aligned} \tag{13}$$

where A is the matrix of elements

$$A = \int_0^1 \bar{f}_x(\tau, v+\alpha\Delta v)\, d\alpha. \tag{14}$$

The second members of the associated set of differential equations

$$\frac{dl}{d\tau} = -lA = g(\tau, l, \tau_0, x_0) \tag{15}$$

* See, for example, Jordan, *Cours d'analyse*, vol. 1, p. 247.

are linear in the elements of l, and the elements of the matrix A are functions of τ and the variable parameters τ_0, x_0. Furthermore for each fixed pair $f, \bar{f}$ the elements of A are continuous and such that mod $A \leqq \kappa$ in the domain of elements satisfying the conditions

$$\tau_1 \leqq \tau \leqq \tau_2, \qquad (\tau_0, x_0) \text{ in } C_{\delta/2}.$$

The equations (15) are therefore of the form considered in the last paragraph of Section 1 with a replaced by τ_0, x_0, and $\mathfrak{A}$ by $C_{\delta/2}$. Hence they have a matrix $L[\tau, \tau', A, \tau_0, x_0]$ whose rows are solutions of the equations (15), reducing to the identity matrix I for the special values $\tau = \tau'$, and having continuity of type I for all sets $[\tau, \tau', A, \tau_0, x_0]$ satisfying the conditions

$$\tau_1 \leqq \tau \leqq \tau_2, \qquad \tau_1 \leqq \tau' \leqq \tau_2, \qquad A \text{ arbitrary}, \qquad (\tau_0, x_0) \text{ in } C_{\delta/2}.$$

Since in the particular case when A has the value (14) its elements have continuity of type II in the domain of elements satisfying the conditions (12), it follows that L may be regarded as having the form $L[\tau', \tau, \tau_0, x_0, f, \bar{f}]$ with continuity of type II in the same domain.

If the two members of the equation (13) are multiplied by the matrix L it follows with the help of equation (15) that

$$L \frac{d}{d\tau} \Delta v = -\frac{dL}{d\tau} \Delta v + L \Delta f(\tau, v),$$

and integration of this equation from τ_0 to τ' gives

$$[\Delta v]^{\tau=\tau'} = \int_{\tau_0}^{\tau'} L[\tau, \tau', \tau_0, x_0, f, \bar{f}] \Delta f(\tau, v)\, d\tau$$

since at $\tau = \tau'$, $L = I$, and at $\tau = \tau_0$, $\Delta v = 0$. If the variables τ' and τ are interchanged in this equation it takes exactly the form (11) which was to be deduced.

The expression (11) is called a difference function of $v[\tau, \tau_0, x_0, f]$ with respect to f. It is easy to see that v has a difference function of a similar sort with respect to all of its arguments. For let

$$\Delta_1 v = v[\bar{\tau}, \bar{\tau}_0, \bar{x}_0, \bar{f}] - v[\tau, \tau_0, x_0, \bar{f}],$$

$$\Delta_2 v = v[\tau, \tau_0, x_0, \bar{f}] - v[\tau, \tau_0, x_0, f],$$

$$\Delta v = \Delta_1 v + \Delta_2 v,$$

and let $\Delta\tau = \bar{\tau} - \tau$, with similar interpretations for $\Delta\tau_0$, Δx_0, Δf. Then since the functions v have continuous partial derivatives with respect to τ, τ_0, x_0 it follows that by an application of Taylor's formula with integral form of the remainder term to $\Delta_1 v$, and an application of formula (11) to $\Delta_2 v$, that

$$\Delta v = a\Delta\tau + b\Delta\tau_0 + C\Delta x_0 + \int_{\tau_0}^{\tau} L\Delta f d\tau', \tag{16}$$

where

$$a = \int_0^1 v_\tau \, d\alpha, \qquad b = \int_0^1 v_{\tau_0} \, d\alpha, \qquad C = \int_0^1 v_{x_0} \, d\alpha, \tag{17}$$

the arguments of the derivatives v_τ, v_{τ_0}, v_{x_0} being $\tau + \alpha\Delta\tau$, $\tau_0 + \alpha\Delta\tau_0$, $x_0 + \alpha\Delta x_0$, $\bar{f}$. The elements of a, b, C have clearly continuity of type II for all elements $[\bar{\tau}, \bar{\tau}_0, \bar{x}_0, \bar{f}]$ and $[\tau, \tau_0, x_0, f]$ in R since the derivatives of v have this property.

The difference

$$\Delta v = v[\tau, \bar{\tau}_0, \bar{x}_0, \bar{f}] - v[\tau, \tau_0, x_0, f]$$

is expressible in the form (16), *in which* a, b, C, L *have elements which are functions of* τ' *and the elements of the sets* $[\bar{\tau}, \bar{\tau}_0, \bar{x}_0, \bar{f}]$ *and* $[\tau, \tau_0, x_0, f]$, *and which have continuity of type II when these variables satisfy the conditions* (12).

It is interesting to note that when the functions v have difference functions of the type (16), they will also have *differentials* in a sense analogous to that defined by Fréchet in the theory of functions of lines. In the first place, when $\bar{\tau} = \tau$, $\bar{\tau}_0 = \tau_0$, $\bar{x}_0 = x_0$, $\bar{f} = f$, the values of the expressions (17) are simply $a = v_\tau$, $b = v_{\tau_0}$, $C = v_{x_0}$. Furthermore if L' denotes the value of L for the same arguments, and if δv is used to represent the expression

$$\delta v = v_\tau \Delta\tau + v_{\tau_0}\Delta\tau_0 + v_{x_0}\Delta x_0 + \int_{\tau_0}^{\tau} L' \Delta f d\tau' \tag{18}$$

then it follows with the help of equation (16) that

$$\Delta v = \delta v + \mu h[\Delta\tau, \Delta\tau_0, \Delta x_0, \Delta f] \tag{19}$$

where μ is the maximum of $|\Delta\tau|$, $|\Delta\tau_0|$, mod Δx_0, mod Δf, mod Δf_x in C_ϵ, and h is a multi-partite function whose modulus approaches zero with μ. For the value of h is in fact

$$h = (a - v_\tau)\frac{\Delta\tau}{\mu} + (b - v_{\tau_0})\frac{\Delta\tau_0}{\mu} + (C - v_{x_0})\frac{\Delta x_0}{\mu} + \int_{\tau_0}^{\tau} (L - L')\frac{\Delta f}{\mu} d\tau',$$

and the modulus of this approaches zero with μ since $|\Delta\tau/\mu|$, $|\Delta\tau_0/\mu|$, mod $(\Delta x_0/\mu)$, mod $(\Delta f/\mu)$ are at most equal to unity, and a, b, C, L approach continuously the limiting values v_τ, v_{τ_0}, v_{x_0}, L' when the elements of $[\bar{\tau}, \bar{\tau}_0, \bar{x}_0, \bar{f}, \bar{f}_x]$ approach those of $[\tau, \tau_0, x_0, f, f_x]$.

The expression (18) *is called the differential of* v *at the set* $[\tau, \tau_0, x_0, f]$. *It gives the value of* v *to terms of the first order in the variables* $\Delta\tau$, $\Delta\tau_0$, Δx_0, Δf *as described above. The matrix* L' *is the matrix whose rows are solutions of the differential equations*

$$\frac{dl}{d\tau'} = -lf_x(\tau', v[\tau', \tau_0, x_0, f]) \tag{20}$$

with the initial values $L = I$ *at* $\tau' = \tau$.

The last statement of this theorem is true because the second members of equations (15) reduce to those of (20) when $[\bar{\tau}, \bar{\tau}_0, \bar{x}_0, \bar{f}]$ and $[\tau, \tau_0, x_0, f]$ coincide.

It is useful to note that the differences Δv are infinitesimals of the first order at least with respect to the variable μ, and that the quotients $\Delta v/\mu$ remain finite as μ approaches zero. This follows readily by dividing through the expression (16) for Δv by μ and remembering that the quotients $\Delta\tau/\mu$, $\Delta\tau_0/\mu$, $\Delta x_0/\mu$, $\Delta f/\mu$ are all bounded as μ approaches zero.

3. Theorems concerning implicit functions

Consider an equation of the form

$$\phi[\tau, \tau_0, x_0, f] = 0$$

having an initial solution $[\tau', \tau_0', x_0', f']$, and let $[\tau', \tau_0', x_0', f']_\epsilon$ denote the totality of elements $[\tau, \tau_0, x_0, f]$ such that

$$|\tau - \tau'|, \ |\tau_0 - \tau_0'|, \ \text{mod}\,(x_0 - x_0'), \ \text{mod}\,(f - f') \leqq \epsilon.$$

Then it is possible to prove the following theorem:

Theorem. *The function* ϕ *and its derivative* ϕ_τ *are supposed to have continuity of type I in a neighborhood* $[\tau', \tau_0', x_0', f']_\epsilon$, *and it is furthermore presupposed that*

$$\phi[\tau', \tau_0', x_0', f'] = 0, \qquad \phi_\tau[\tau', \tau_0', x_0', f'] \neq 0.$$

Then it will be true that

(1) *the constant* ϵ *can be so restricted that in the neighborhood* $[\tau', \tau_0', x_0', f']_\epsilon$ *no two solutions* $[\tau_1, \tau_0, x_0, f]$, $[\tau_2, \tau_0, x_0, f]$ *of the equation* $\phi = 0$ *exist with* $\tau_1 \neq \tau_2$;

(2) *there is a neighborhood* $[\tau_0', x_0', f']_\delta$ *with every element of which there is associated a solution* $[\tau, \tau_0, x_0, f]$ *in* $[\tau', \tau_0', x_0', f']_\epsilon$;

(3) *the function* $\tau[\tau_0, x_0, f]$ *so defined has continuity of type I in* $[\tau_0', x_0', f']_\delta$.

It is an immediate consequence of the theorem that $\tau' = \tau[\tau_0', x_0', f']$.

To prove the theorem it may first be remarked that on account of the continuity of the derivative ϕ_τ the neighborhood $[\tau', \tau_0', x_0', f']_\epsilon$ can be so restricted that $\phi_\tau \neq 0$ everywhere in it. The conclusion (1) of the theorem is justified in this neighborhood since ϕ is there a monotonic function of τ vanishing at most once in the interval $|\tau - \tau'| \leqq \epsilon$ when τ_0, x_0, f are fixed.

The values $\phi[\tau' + \epsilon, \tau_0', x_0', f']$ and $\phi[\tau' - \epsilon, \tau_0', x_0', f']$ have now opposite signs, and on account of the continuity of ϕ it is possible to choose $\delta \leqq \epsilon$

so small that $\phi[\tau' + \epsilon, \tau_0, \xi_0, f]$ and $\phi[\tau' - \epsilon, \tau_0, x_0, f]$ will also have opposite signs for every $[\tau_0, x_0, f]$ in the neighborhood $[\tau_0', x_0', f']_\delta$. For every such $[\tau_0, x_0, f]$ the function $\phi[\tau, \tau_0, x_0, f]$ will be a monotonic function of τ and will vanish at one value on the interval $\tau' - \epsilon \leqq \tau \leqq \tau' + \epsilon$, which proves the conclusion (2) of the theorem.

For two solutions $[\tau, \tau_0, x_0, f]$, $[\tau + \Delta\tau, \tau_0 + \Delta\tau_0, x_0 + \Delta x_0, f + \Delta f]$ with projections in the neighborhood $[\tau_0', x_0', f']_\delta$ the equations

$$\begin{aligned} 0 &= \phi[\tau + \Delta\tau, \tau_0 + \Delta\tau_0, x_0 + \Delta x_0, f + \Delta f] \\ &\qquad - \phi[\tau, \tau_0 + \Delta\tau_0, x_0 + \Delta x_0, f + \Delta f] \\ &\quad + \phi[\tau, \tau_0 + \Delta\tau_0, x_0 + \Delta x_0, f + \Delta f] - \phi[\tau, \tau_0, x_0, f] \\ &= \phi_1 \Delta\tau + \phi[\tau, \tau_0 + \Delta\tau_0, x_0 + \Delta x_0, f + \Delta f] - \phi[\tau, \tau_0, x_0, f] \end{aligned}$$

are true, where

$$\phi_1 = \int_0^1 \phi_\tau[\tau + \alpha\Delta\tau, \tau_0 + \Delta\tau_0, x_0 + \Delta x_0, f + \Delta f]\, d\alpha.$$

Since ϕ is continuous and $\phi_1 \neq 0$ it follows that $\Delta\tau$ approaches zero with $|\Delta\tau_0|$, mod Δx_0, mod Δf, and hence that the function $\tau[\tau_0, x_0, f]$ is continuous in $[\tau_0', x_0', f']_\delta$, as stated in conclusion (3) of the theorem.

If the function ϕ is one of the functions v of the preceding section, the solution $\tau[\tau_0, x_0, f]$ of the equation $\phi = 0$ will have a difference function

$$\Delta\tau = -\frac{\beta}{\alpha}\Delta\tau_0 - \frac{c}{\alpha}\Delta x_0 - \frac{1}{\alpha}\int_{\tau_0}^{\tau} l\Delta f d\tau', \tag{21}$$

where α, β, c, l represent those of the elements of a b, C, L in equation (16) *which pertain to the particular function ϕ. In them the arguments τ, $\bar{\tau}$ are supposed replaced by the functions $\tau[\tau_0, x_0, f]$, $\tau[\bar{\tau}_0, \bar{x}_0, \bar{f}]$.*

This is readily provable because the equation

$$\Delta\phi = \alpha\Delta\tau + \beta\Delta\tau_0 + c\Delta x_0 + \int_{\tau_0}^{\tau} l\Delta f d\tau'$$

holds for arbitrary arguments $[\tau, \tau_0, x_0, f]$, $[\bar{\tau}, \bar{\tau}_0, \bar{x}_0, \bar{f}]$, and these sets make both members of the equation vanish when $\tau = \tau[\tau_0, x_0, f]$ and $\bar{\tau} = \tau[\bar{\tau}_0, \bar{x}_0, \bar{f}]$, since then both of the sets are solutions of the equation $\phi = 0$.

If ϕ and ψ are both particular functions of the set v of the preceding sections, and if the solution $\tau[\tau_0, x_0, f]$ of the equation $\phi = 0$ is substituted in ψ, then the resulting function ψ of τ_0, x_0, f has a difference function

$$\Delta_1 \psi = -\frac{\bar{\alpha}}{\alpha}\left(\beta\Delta\tau_0 + c\Delta x_0 + \int_{\tau_0}^{\tau} l\Delta f d\tau'\right)$$

$$+ \bar{\beta}\Delta\tau_0 + \bar{c}\Delta x_0 + \int_{\tau_0}^{\tau} \bar{l}\Delta f d\tau',$$

where $\bar{\alpha}, \bar{\beta}, \bar{c}, \bar{l}$ *are symbols for those elements of* a, b, C, L *in equations* (16) *which belong to* ψ, *with the arguments* $\tau = \tau[\tau_0, x_0, f]$, $\bar{\tau} = \tau[\bar{\tau}_0, \bar{x}_0, \bar{f}]$.

This result is found by substituting the expression (21) for $\Delta\tau$ in the difference function

$$\Delta\psi = \bar{\alpha}\Delta\tau + \bar{\beta}\Delta\tau_0 + \bar{c}\Delta x_0 + \int_{\tau_0}^{\tau} \bar{l}\Delta f d\tau'$$

of the function ψ. The differential $\delta_1\psi$ of ψ with respect to the arguments τ_0, x_0, f after the substitution $\tau = \tau[\tau_0, x_0, f]$ is evidently

$$\delta_1 \psi = -\frac{\psi_\tau}{\phi_\tau}\left(\phi_{\tau_0}\Delta\tau_0 + \phi_{x_0}\Delta x_0 + \int_{\tau_0}^{\tau} l'\Delta f d\tau'\right)$$

$$+ \psi_{\tau_0}\Delta\tau_0 + \psi_{x_0}\Delta x_0 + \int_{\tau_0}^{\tau} \bar{l}'\Delta f d\tau',$$

where the arguments of the derivatives of ϕ, ψ are $\tau[\tau_0, x_0, f], \tau_0, x_0, f$.

FUNCTIONS OF LINES IN BALLISTICS*

BY

GILBERT AMES BLISS

In the theory of ballistics it is customary first of all to set up the differential equations describing the motion of a projectile under the assumptions that no wind is blowing and that the densities of the air at different altitudes are normal. If the initial velocity and the initial direction of flight of the projectile are given, it is then possible to compute the coördinates of the projectile as functions of the time t satisfying these differential equations. The path of the projectile is different, however, when a wind is blowing or when the air densities are not normal, and the form of the differential equations themselves must be changed to account for such disturbed conditions. The new differential equations involve the velocity $w(y)$ of the wind in the direction of the plane of fire at the altitude y, and the correction $\kappa(y)$ for the density at the altitude y. The range X thus becomes a function $X[w(y), \kappa(y)]$ of the functions $w(y)$, $\kappa(y)$, or in other words a function of lines, and the first differential of X in the sense of the theory of functions of lines is the first order effect of wind and density upon the range.

The range is the quantity of greatest interest to the ballistician in most problems of artillery fire. But for anti-aircraft guns one must also investigate the corrections to the three coördinates x, y, z of the projectile at the ends of successive short intervals of time t along the trajectory. It is clear that these coördinates will also be functions of lines in the sense described above for the range X.

In the following pages a method for computing the differentials of such functions of lines is described. The formulas found are the results of applications, to differential equations of the forms used in ballistics, of more general results deduced by the writer in a preceding paper.† The proofs here given, however, have been made independent as far as possible of those for the general case. The methods used have proved to be of value in the practical computations of the differential corrections hitherto deemed necessary for trajectories, and as the applications of the theory of ballistics become more

* Presented to the Society, September, 1919.

† *Differential equations containing arbitrary functions*, these Transactions, vol. 21 (1920), p. 79.

refined the formulas will very likely be of value in the computation of other corrections also. The writer has already published accounts of the method with such mathematical details as are necessary for the practical ballistician.* The purpose of the present paper is to show the connection between the ballistic theory and the theory of functions of lines. The method given for finding the first differential of a function of lines is one which could be applied to many cases of such functions defined by differential equations.

1. The fundamental formula for differential corrections

Consider a set of differential equations of the form

$$\begin{aligned} x'' &= f(t, x, y, z, x', y', z'), \\ y'' &= g(t, x, y, z, x', y', z'), \\ z'' &= h(t, x, y, z, x', y', z'), \end{aligned} \tag{1}$$

and having an initial solution

$$(C) \qquad x = x(t), \quad y = y(t), \quad z = z(t) \qquad (t_1 \leqq t \leqq t_2)$$

for which the functions $x(t)$, $y(t)$, $z(t)$ have continuous derivatives of the first and second orders on the interval $t_1 t_2$, and which passes through an initial element $(t_0, x_0, y_0, z_0, x_0', y_0', z_0')$. In a neighborhood N of the values (t, x, y, z, x', y', z') on this solution the functions f, g, h are supposed to have continuous partial derivatives up to and including those of the second order at least.

If the functions f, g, h are replaced by three others $\bar{f}$, $\bar{g}$, $\bar{h}$, having similar properties in the neighborhood N, the solutions of the equations

$$\begin{aligned} \bar{x}'' &= \bar{f}(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}'), \\ \bar{y}'' &= \bar{g}(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}'), \\ \bar{z}'' &= \bar{h}(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}'), \end{aligned} \tag{2}$$

will be uniquely determined by an initial element $(t_0, x_0, y_0, z_0, \bar{x}_0', \bar{y}_0', \bar{z}_0')$, and if t_0, x_0, y_0, z_0 are kept fixed $\bar{x}$ may be regarded as a function $\bar{x}[\bar{x}_0', \bar{y}_0', \bar{z}_0', \bar{f}, \bar{g}, \bar{h}]$ with similar expressions for $\bar{y}$ and $\bar{z}$. It is natural to expect that by taking the elements of the set $[\bar{x}_0', \bar{y}_0', \bar{z}_0', \bar{f}, \bar{g}, \bar{h}]$ sufficiently near to those of the set $[x_0', y_0', z_0', f, g, h]$ belonging to the solution C the three functions

* *A method of computing differential corrections for a trajectory,* Journal of the United States Artillery, vol. 51 (1919), p. 445; *The use of adjoint systems in the problem of differential corrections for trajectories,* ibid., p. 296; *Differential corrections for anti-aircraft guns,* (not yet published).

$\bar{x}$, $\bar{y}$, $\bar{z}$ may be made to differ as little as is desired from the functions $x(t)$, $y(t)$, $z(t)$ defining that solution. That this is the case follows readily from a theorem of the first paper by the writer referred to in the introduction above,* provided that the equations ( ) are written in the normal form

$$\frac{dx}{dt} = x', \qquad \frac{dy}{dt} = y', \qquad \frac{dz}{dt} = z', \qquad \frac{dx'}{dt} = f, \qquad \frac{dy'}{dt} = g, \qquad \frac{dz'}{dt} = h.$$

If Δx denotes the difference $\bar{x} - x$, with similar notations for the other variables, the quotients $\Delta x/m_1$, $\Delta x'/m_1$, and the similar ones for y and z, are all bounded as m_1 approaches zero, m_1 being the maximum of $|\Delta x_0'|$, $|\Delta y_0'|$, $|\Delta z_0'|$, $|\Delta f|$, $|\Delta g|$, $|\Delta h|$.†

By subtracting equations (1) from equations (2) a relation of the form

$$\Delta x'' = f(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}') - f(t, x, y, z, x', y', z') + \Delta f(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}')$$

is found for Δx, and similar ones hold for Δy and Δz. This equation may also be written in the form

$$(3) \quad \Delta x'' = f_x \Delta x + f_y \Delta y + f_z \Delta z + f_{x'} \Delta x' + f_{y'} \Delta y' + f_{z'} \Delta z' + \Delta f + \phi,$$

where

$$\phi = f(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}') - f - f_x \Delta x - f_y \Delta y - f_z \Delta z - f_{x'} \Delta x' - f_{y'} \Delta y' - f_{z'} \Delta z' + \Delta f(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}') - \Delta f.$$

Similar equations hold for Δy and Δz with f, ϕ replaced by g, ψ and h, χ. It is understood that arguments not indicated in f, g, h and their derivatives are always those belonging to the curve C. By applications of Taylor's formula with remainder terms of the second degree‡ to the first eight terms of the expression for ϕ, and also to the last two, it is clear that ϕ is of the second degree in Δx, Δy, Δz, $\Delta x'$, $\Delta y'$, $\Delta z'$ and the derivatives Δf_x, Δf_y, Δf_z, $\Delta f_{x'}$, $\Delta f_{y'}$, $\Delta f_{z'}$. If m_2 is the maximum of m_1 and $|\Delta f_x|$, $|\Delta f_y|$, $|\Delta f_z|$, $|\Delta f_{x'}|$, $|\Delta f_{y'}|$, $|\Delta f_{z'}|$, then ϕ/m_2^2 is bounded as m_2 approaches zero, and $\phi = m_2^2(\phi/m_2^2)$ itself may therefore be regarded as being of the second degree in m_2, as will also be the case for ψ and χ.

Equation (3) and the similar ones for $\Delta y''$ and $\Delta z''$ may be written in the normal form

* Loc. cit., p. 85.

† Loc. cit., p. 89.

‡ The formula with integral form of the remainder terms is the most satisfactory. See Jordan, *Cours d'analyse*, vol. 1, 2d edition, p. 247.

$$
\begin{aligned}
\frac{d}{dt}\Delta x &= \Delta x', \\
\frac{d}{dt}\Delta y &= \Delta y', \\
\frac{d}{dt}\Delta z &= \Delta z', \\
\frac{d}{dt}\Delta x' &= f_x\Delta x + f_y\Delta y + f_z\Delta z + f_{x'}\Delta x' + f_{y'}\Delta y' + f_{z'}\Delta z' + \Delta f + \phi, \\
\frac{d}{dt}\Delta y' &= g_x\Delta x + g_y\Delta y + g_z\Delta z + g_{x'}\Delta x' + g_{y'}\Delta y' + g_{z'}\Delta z' + \Delta g + \psi, \\
\frac{d}{dt}\Delta z' &= h_x\Delta x + h_y\Delta y + h_z\Delta z + h_{x'}\Delta x' + h_{y'}\Delta y' + h_{z'}\Delta z' + \Delta h + \chi.
\end{aligned} \tag{4}
$$

The system adjoint to this is by definition the system

$$
\begin{aligned}
-\lambda' &= f_x\rho + g_x\sigma + h_x\tau, \\
-\mu' &= f_y\rho + g_y\sigma + h_y\tau, \\
-\nu' &= f_z\rho + g_z\sigma + h_z\tau, \\
-\rho' &= \lambda + f_{x'}\rho + g_{x'}\sigma + h_{x'}\tau, \\
-\sigma' &= \mu + f_{y'}\rho + g_{y'}\sigma + h_{y'}\tau, \\
-\tau' &= \nu + f_{z'}\rho + g_{z'}\sigma + h_{z'}\tau,
\end{aligned} \tag{5}
$$

formed by using the columns of coefficients of the system (4), with signs changed, in place of the rows. If $\lambda, \mu, \nu, \rho, \sigma, \tau$ is a solution of the last system, then by multiplying the equations of the system (4) by them, respectively, and using equations (5), it follows that

$$
\begin{aligned}
\frac{d}{dt}(\lambda\Delta x + \mu\Delta y + \nu\Delta z + \rho\Delta x' + \sigma\Delta y' + \tau\Delta z') \\
= \rho\Delta f + \sigma\Delta g + \tau\Delta h + \rho\phi + \sigma\psi + \tau\chi.
\end{aligned}
$$

Furthermore by integrating this last result from t_0 to t, the relation

$$
\begin{aligned}
&\lambda\Delta x + \mu\Delta y + \nu\Delta z + \rho\Delta x' + \sigma\Delta y' + \tau\Delta z' \\
&\quad = [\lambda\Delta x + \mu\Delta y + \nu\Delta z + \rho\Delta x' + \sigma\Delta y' + \tau\Delta z']^{t=t_0} \\
&\qquad + \int_{t_0}^{t} (\rho\Delta f + \sigma\Delta g + \tau\Delta h)\,dt + \int_{t_0}^{t} (\rho\phi + \sigma\psi + \tau\chi)\,dt
\end{aligned} \tag{6}
$$

is found.

In his last equation the second integral is of the second degree in the maximum m_2, since ϕ, ψ, χ have this property, as explained above. The first two terms on the right may therefore be regarded as the differential of the sum on the left with respect to the differences $\Delta x_0'$, $\Delta y_0'$, $\Delta z_0'$, Δf, Δg, Δh. If t is considered as fixed for the moment, and if $(\lambda, \mu, \nu, \rho, \sigma, \tau)$ are so chosen that their initial values at the particular value t are $(1, 0, 0, 0, 0, 0)$, then the formula (6) gives an expression for Δx in which the first order terms are the first differential δx of the function $\bar{x}$ in terms of the differences $\Delta x_0'$, $\Delta y_0'$, $\Delta z_0'$, Δf, Δg, Δh. In a similar manner the differentials of $\bar{y}$ and $\bar{z}$ are determined. It is justifiable then to deduce a differential formula from (6) which takes the form

$$\begin{aligned}(7)\qquad &\lambda\delta x + \mu\delta y + \nu\delta z + \rho\delta x' + \sigma\delta y' + \tau\delta z' \\ &\qquad = [\rho\Delta x_0' + \sigma\Delta y_0' + \tau\Delta z_0']^{t=t_0} + \int_{t_0}^{t} (\rho\Delta f + \sigma\Delta g + \tau\Delta h)\, dt.\end{aligned}$$

It is the equation which expresses the equality of the first order terms in equation (6). The existence of the differentials of $\bar{x}$, $\bar{y}$, $\bar{z}$, and their derivatives is also an immediate consequence of the theorems of the paper referred to above.*

The last equation is fundamental for what follows. It gives the first order terms with respect to the variables $\Delta x_0'$, $\Delta y_0'$, $\Delta z_0'$, Δf, Δg, Δh of every expression of the form

$$(8)\qquad \lambda\Delta x + \mu\Delta y + \nu\Delta z + \rho\Delta x' + \sigma\Delta y' + \tau\Delta z'$$

for which $(\lambda, \mu, \nu, \rho, \sigma, \tau)$ is a solution of the adjoint system (5), and it may be called the first differential of such an expression with respect to the variables x_0', y_0', z_0' and the functions f, g, h. If the first order terms are desired at a fixed value of t for an expression of the type (8) with arbitrarily chosen coefficients, it is only necessary to insert in the equation (7) a set of solutions $\lambda, \mu, \nu, \rho, \sigma, \tau$ of equations (5) with initial values at t equal to the coefficients in the expression (8). On the other hand, if t is variable, then six linearly independent solutions of equations (5) give six independent equations of the type (7) from which the six differentials δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$ can be calculated. It is to be explained in the following sections how these considerations can be applied to calculate the differential corrections for trajectories.

2. Formulas for range corrections and deflections in ballistics

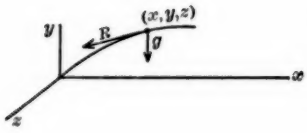
The differential equations of the trajectory of a projectile have the form

* Loc. cit., p. 89.

$$x'' = -R\frac{x'}{v} = -Fx',$$

$$y'' = -R\frac{y'}{v} - g = -Fy' - g, \tag{9}$$

$$z'' = -R\frac{z'}{v} = -Fz',$$



where R is the retardation of the projectile in the direction of the tangent due to the resistance of the air, g is the gravitational constant, and F merely a convenient notation for R/v. The value of R used by Moulton is

$$R = vF(v, y) = v\frac{G(v)e^{-ay}}{C}, \qquad v = \sqrt{x'^2 + y'^2 + z'^2},$$

where $vG(v)$ is the resistance of the air to a standard projectile moving at sea level with velocity v, C is the ballistic coefficient which enables the formula to be fitted to projectiles of other types, and e^{-ay} is the altitude function which accounts for the diminution in the retardation as the altitude y increases and the density of the air diminishes. The values of $G(v)$ have been tabulated for different velocities from observations of actual firings and the table has been smoothed by means of an analytic function which agrees with the observed data with sufficient accuracy. The value of the constant $a = .000106$ for use with the meter-second system of units, is the result of meteorological observations extending over many years. It is so chosen that the function e^{-ay} represents the ratio of the density of the air at the altitude y to the density of the air at sea level under so-called normal or average conditions. These normal conditions rarely exist at an actual time of firing, and the variations from them must be taken into account as will be explained in the following paragraph.

It is presupposed that a trajectory

$$x = x(t), \qquad y = y(t), \qquad z = 0 \qquad (0 \leqq t \leqq T) \tag{10}$$

has been computed with the help of equations (9), having given initial values $x = y = z = 0$, $x' = x'_0$, $y' = y'_0$, $z' = 0'_0$ at the time $t = 0$. The values of z will all be zero since the equations (9) have one and but one solution for the prescribed initial values, and since such a solution can be found by setting $z = 0$ in the equations and integrating the first two for x and y. The time of flight of the projectile will be denoted by T and the range by X. The actual computation may be most economically effected in a particular case by the methods of approximation devised by F. R. Moulton and his associates in

the ballistic work at Washington,* the result being a table of values of the functions (10) at the ends of short intervals of time from $t = 0$ to $t = T$. The problem of the present paper is then the determination of the corrections which must be applied to the values of these functions, and in particular to the range X, in order to account for disturbances of the trajectory of various sorts, some of which will be explained below.

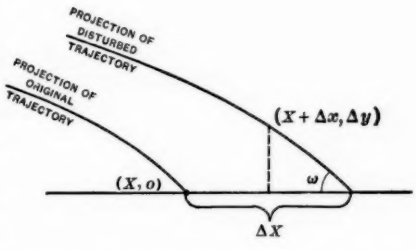
If the second members of equations (9) are denoted by f, g, h, and if they are replaced in any way by new functions

$$\bar{f} = f + \Delta f, \qquad \bar{g} = g + \Delta g, \qquad \bar{h} = h + \Delta h,$$

in order to account for disturbances of the trajectory, then the results described in Section 1 will be applicable. The linear combinations (7) of greatest interest to the ballistician are the deflection $\delta z(T)$ and the differential δX of the range X. To find a first expression for the latter one may proceed intuitively as follows. The coördinates of the projectile on the disturbed trajectory at the time T are $(X + \Delta x, \Delta y, \Delta z)$ and the value

$$\Delta X = [\Delta x + \cot \omega \Delta y]^{t=T},$$

where ω is the positive angle of fall, is approximately correct for ΔX, as one sees readily from the figure which shows the projections of the original and the disturbed trajectory on the plane of fire. The value of the differential δX is found by replacing the differences Δx and Δy by differentials in this result.



A more satisfactory method of determining δX from the standpoint of the mathematical analyst is, however, to apply the theorems on implicit functions of the paper referred to above.† The equation

$$\bar{y}[\bar{T}, \bar{x}_0', \bar{y}_0', \bar{z}_0', \bar{f}, \bar{g}, \bar{h}] = 0,$$

where $\bar{y}$ is the function of Section 1 above for the special case of equations (9), has an initial solution $[T, x_0', y_0', 0, f, g, h]$ at the end of the trajectory (10), in a neighborhood of which the implicit function theorem is applicable. The solution $\bar{T}[\bar{x}_0', \bar{y}_0', \bar{z}_0', \bar{f}, \bar{g}, \bar{h}]$ of this equation is the time of flight on the disturbed trajectory and the expression

$$X[\bar{x}_0', \bar{y}_0', \bar{z}_0', \bar{f}, \bar{g}, \bar{h}] = \bar{x}[\bar{T}, \bar{x}_0', \bar{y}_0', \bar{z}_0', \bar{f}, \bar{g}, \bar{h}]$$

* For a sketch of his methods see Journal of the United States Artillery, vol. 51 (1919), p. 40.

† Loc. cit., p. 90.

is the corresponding range. According to the resuits of the paper just cited the differential of X is

$$\delta X = \left[\delta x - \frac{x'}{y'} \delta y \right]^{t=T} = [\, \delta x + \cot \omega \delta y \,]^{t=T}, \tag{11}$$

where δx and δy are the differentials of $\bar{x}$ and $\bar{y}$ with respect to $\bar{x}_0'$, $\bar{y}_0'$, $\bar{z}_0'$, $\bar{f}$, $\bar{g}$, $\bar{h}$.

The differential equations corresponding to the system (5) for the special case of this section are readily found to be

$$\begin{aligned} \lambda' &= 0, \\ \mu' &= - ax' F\rho - ay' F\sigma, \\ \nu' &= 0, \\ \rho' &= - \lambda + F\left(1 + x'^2 \frac{G'}{vG}\right)\rho + x' y' F \frac{G'}{vG}\sigma, \\ \sigma' &= - \mu + x' y' F \frac{G'}{vG}\rho + F\left(1 + y'^2 \frac{G'}{vG}\right)\sigma, \\ \tau' &= - \nu + F\tau. \end{aligned} \tag{12}$$

To find δX from formulas (7) and (11) a solution $(\lambda, \mu, \nu, \rho, \sigma, \tau)$ of this system with initial values $(1, \cot \omega, 0, 0, 0, 0)$ at $t = T$ is needed. For such a solution it is clear that $\nu \equiv \tau \equiv 0$ so that the equation (7) gives

$$\delta X = \rho(0)\Delta x_0' + \sigma(0)\Delta y_0' + \int_0^T (\rho \Delta f + \sigma \Delta g)\, dt. \tag{13}$$

Similarly to find $\delta z(T)$ a solution of equations (12) must be used with initial values $(0, 0, 1, 0, 0, 0)$ at $t = T$. The functions $\lambda, \mu, \rho, \sigma$ are all identically zero in this case since the first, second, fourth, and fifth of equations (12) are linear and homogeneous in $\lambda, \mu, \rho, \sigma$ alone and have only the solutions $\lambda \equiv \mu \equiv \rho \equiv \sigma \equiv 0$ with the initial values $(0, 0, 0, 0)$. Formula (7) then gives

$$\delta z(T) = \tau(0)\Delta z_0' + \int_0^T \tau \Delta h\, dt. \tag{14}$$

The function τ in the last formula can be expressed in terms of the known function $x(t)$ of the trajectory (10). For $F = - x''/x'$, from the first of the equations (9), and consequently the last of equations (12) gives

$$\tau' x' + \tau x'' = - x',$$

since now $\nu \equiv 1$. Integration of this result, and evaluation of the constant of integration by setting $t = T$, gives

$$\tau = \frac{X - x(t)}{x'(t)}. \tag{15}$$

The expressions (13) *and* (14) *are the first differentials of the range* X *and the deflection* $z(T)$ *thought of as functions of the ordinary variables* x_0', y_0', z_0' *and of the functions* f, g, h. *They give respectively the first order range correction, and the first order deflection, corresponding to a disturbance of the trajectory which can be accounted for by altering the initial conditions, or by changing the functions* f, g, h *in the second members of equations* (9) *into* $f + \Delta f$, $g + \Delta g$, $h + \Delta h$, *or both.*

3. Applications to special corrections

The formulas (13) and (14) are applicable to a very great variety of problems. Suppose, for example, that experiments had been devised which gave a new and more accurate table of values of the function $G(v)$ than the one at present used. It would not be necessary to re-compute completely the ranges corresponding under normal conditions to given elevations. For the new value of G could be inserted in equations (9) and the corrections Δf, Δg, Δh computed. The formula (13) would then give first order corrections which could be applied to revise the ranges so that they would be in accord with the new retardation law.

The most important applications of the formulas are, however, in computing the differential corrections due to variations in the initial conditions, variations from normal in the density of the air, or wind. The direction and velocity of the wind at different altitudes is determined in practice by observing the motion of a small balloon through two surveying instruments at the ends of a measured base line, and its components at the altitude y in the plane of fire and at right angles to it will be denoted hereafter by $w_x(y)$ and $w_z(y)$. The variations from normal in the density of the air at different altitudes is determined by means of airplane observations and may be accounted for in equations (9) by using an altitude function of the form $e^{-ay}[1 + \kappa(y)]$ instead of e^{-ay}.

When a wind is blowing the retardation of the projectile is no longer in the line of the tangent to the trajectory, but takes place in the direction of the motion of the projectile with respect to the air, and is determined in magnitude by the velocity V of the projectile with respect to the air. This velocity is the vector sum of the velocity of the earth with respect to the projectile and of the velocity of the air with respect to the earth, and its components and magnitude are

$$-\bar{x}' + w_x(\bar{y}), \qquad -\bar{y}', \qquad -\bar{z}' + w_z(\bar{y}),$$

$$U = \sqrt{[\bar{x}' - w_x(\bar{y})]^2 + \bar{y}'^2 + [\bar{z}' - w_z(\bar{y})]^2},$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ represent the coördinates of the disturbed projectile. The differential equations which determine the trajectory disturbed by wind or abnormal density are now

$$\bar{x}'' = -\bar{F}[\bar{x}' - w_x(\bar{y})],$$

(16) $$\bar{y}'' = -\bar{F}\bar{y}' - g,$$

$$\bar{z}'' = -\bar{F}[\bar{z}' - w_z(\bar{y})],$$

where

(17) $$\bar{F} = \frac{1}{C}G(V)e^{-a\bar{y}}[1 + \kappa(\bar{y})].$$

The solutions of these equations are functions $\bar{x}$, $\bar{y}$, $\bar{z}$ of the arguments t, x_0', y_0', z_0', $w_x(y)$, $w_z(y)$, $\kappa(y)$, and the first order deflection $\delta z(T)$ and range correction δX are given by formulas (14) and (13) when Δf, Δg, Δh are replaced by the first order terms of their expansions in powers of $w_x(y)$, $w_z(y)$, $\kappa(y)$ which are described below.

The second members of equations (16) are the functions previously designated by $\bar{f}$, $\bar{g}$, $\bar{h}$, and the differences $\Delta f = \bar{f} - f$, Δg, Δh can be readily calculated. In the formula (7) their arguments are the elements t, x, y, z, x', y', z' of the known solution C, and in the formulas (13) and (14) it is to be remembered that they must similarly be the analogous elements of the solution (10) already computed. It follows then that to first order terms in $w_x(y)$, $w_z(y)$, $\kappa(y)$

$$\Delta f = F\left(1 + x'^2\frac{G'}{vG}\right)w_x(y) - x'F\kappa(y) + \cdots,$$

$$\Delta g = x'y'F\frac{G'}{vG}w_x(y) - y'F\kappa(y) + \cdots,$$

$$\Delta h = Fw_z(y) + \cdots.$$

When these values are substituted in (13) and (14), and use is made of the equations (12) and (15), it is found that

(18) $$\delta X \doteq \rho(0)\Delta x_0' + \sigma(0)\Delta y_0' + \int_0^T(\rho' + 1)w_x(y)\,dt + \frac{1}{a}\int_0^T \mu'\kappa(y)\,dt,$$

(19) $$\delta z(T) = \frac{X}{x_0'}\Delta z_0' + \int_0^T(\tau' + 1)w_z(y)\,dt.$$

The expressions (18) *and* (19) *are the first differentials of the range* X *and the deflection* $z(T)$ *thought of as functions of the ordinary variables* x_0', y_0', z_0' *and of the wind- and density-functions* $w_x(y)$, $w_z(y)$, $\kappa(y)$. *They give, respectively, the first order range correction and the deflection due* (1) *to a variation in the initial velocity with projections* $\Delta x_0'$, $\Delta y_0'$, $\Delta z_0'$ *on the coördinate axes,* (2) *to a wind at the altitude* y *with components* $w_x(y)$, $w_z(y)$ *in the plane of fire and at right angles to it, and* (3) *to a variation from normal in the density ratio of the air which at altitude* y *is equal to the fraction* $\kappa(y)$ *of the normal*

density ratio e^{-ay}. *The functions* $\lambda, \mu, \rho, \sigma$ *occurring in these equations are solutions of the first, second, fourth, and fifth of equations* (12) *with initial values* $(1, \cot \omega, 0, 0)$ *at* $t = T$, ω *being the positive angle of fall. The function* τ *is given by the equation* (15).

In constructing a range table it is sometimes desired to compute the first order correction to the range X due to the use of a slightly different value $C + \Delta C$ of the ballistic coefficient. The formula (18) will give this correction also, if the function $\kappa(y)$ is everywhere replaced by $\kappa(y) - \Delta C/C$. For the equations (16) still hold provided that the expression (17) for $\bar{F}$ is multiplied by the factor

$$\frac{1}{1 + \dfrac{\Delta C}{C}} = 1 - \frac{\Delta C}{C} + \cdots,$$

which gives, to first order terms, the value

$$\bar{F} = \frac{1}{C} G(V) e^{-ay} \left[1 + \kappa(y) - \frac{\Delta C}{C} + \cdots \right].$$

This is the same as before except that $\kappa(y)$ is replaced by $\kappa(y) - \Delta C/C$.

The versatility of the formulas (13) and (14) of Section 2 is still further illustrated by the problem of determining the effect of the rotation of the earth upon a trajectory. Roever has set up the differential equations of motion in the form

$$\begin{aligned} \bar{x}'' &= -\bar{F}\bar{x}' + 2\omega(\bar{y}' \cos\phi \sin\alpha - \bar{z}' \sin\phi), \\ \bar{y}'' &= -\bar{F}\bar{y}' - g - 2\omega \cos\phi(\bar{z}' \cos\alpha + \bar{x}' \sin\alpha), \\ \bar{z}'' &= -\bar{F}\bar{z}' + 2\omega(\bar{x}' \sin\phi + \bar{y}' \cos\phi \cos\alpha). \end{aligned} \tag{20}$$

In these formulas ϕ is the latitude of the gun position, α is the azimuth of the plane of fire measured clockwise from the south, $\omega = .0000729$ is the angular velocity of the earth's rotation in radians per second, and $\bar{F}$ is the value (17) except that now $w_x(y) = w_z(y) = \kappa(y) = 0$, since only the effect of the rotation of the earth apart from other disturbances is to be considered. The effects of all the disturbances could be evaluated simultaneously if desired but the result would be merely to add the terms about to be deduced to the expressions (18) and (19). In the equations (20) powers of ω higher than the first have been neglected.

The second member of the first of equations (20) is now the function $\bar{f}(t, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}')$, and the value of Δf in the formula (13) is the difference $\bar{f} - f$ formed for the arguments t, x, y, z, x', y', z' of the initial trajectory (10), with similar values for Δg and Δh. It follows readily that

$$\Delta f = 2\omega y' \cos\phi \sin\alpha,$$

$$\Delta g = -2\omega x' \cos\phi \sin\alpha,$$

$$\Delta h = 2\omega (x' \sin\phi + y' \cos\phi \cos\alpha),$$

and the formulas (13) and (14) give

$$\delta X = 2\omega \cos\phi \sin\alpha \int_0^T (\rho y' - \sigma x')\, dt, \tag{21}$$

$$\delta z(T) = 2\omega \int_0^T \tau (x' \sin\phi + y' \cos\phi \cos\alpha)\, dt. \tag{22}$$

These are the first differentials of the range X and the deflection $z(T)$ considered as functions of the angular velocity ω of the rotation of the earth. They give the first order effects of the rotation of the earth upon X and $z(T)$.

The advantage, from the standpoint of the computer, of the formulas for the corrections derived in this section lies in the fact that only two integrations of systems of differential equations by methods of approximation have to be made in order to get all of the corrections. The first integration is of the system (9) in order to find the original trajectory (10), and the second is the integration of the first, second, fourth, and fifth of equations (12) for the functions $(\lambda, \mu, \rho, \sigma)$ with initial values $(1, \cot\omega, 0, 0)$ at $t = T$. When this last set of functions has been determined the formulas (18), (19), (21), (22) give the values of the desired corrections by elementary operations of arithmetic, except in the case of the rotation of the earth for which two quadratures by methods of approximation are required. The details of the computation are described in the papers mentioned above* and need not be reproduced here. The methods in use when these results were obtained involved a separate approximate integration of a system of differential equations for each type of correction, and this multiplicity of integrations is now replaced by two, the amount of computational work required being diminished by about three fourths.

4. The differentials of the coördinates x, y, z for anti-aircraft corrections

The details of the computation of anti-aircraft corrections have been described by the writer in another paper.† Theoretically the steps used to obtain them are the following.

A fundamental system of four solutions $(\lambda_i, \mu_i, \rho_i, \sigma_i)$ $(i = 1, \cdots, 4)$ of the first, second, fourth, and fifth of equations (12) gives when substituted in equation (7) four linear relations

*Journal of the United States Artillery, vol. 51 (1919), p. 296.

† Submitted for publication to the Journal of the United States Artillery.

$$\lambda_i\,\delta x + \mu_i\,\delta y + \rho_i\,\delta x' + \sigma_i\,\delta y' = m_i \qquad (i = 1, \cdots, 4). \tag{23}$$

If the fundamental system chosen is the one whose elements reduce to the identity matrix at $t = 0$, the four values m_i from the second members of equation (7) are

$$m_1 = \int_0^t (\rho_1\,\Delta f + \sigma_1\,\Delta g)\,dt, \qquad m_2 = \int_0^t (\rho_2\,\Delta f + \sigma_2\,\Delta g)\,dt,$$

$$m_3 = \Delta x_0' + \int_0^t (\rho_3\,\Delta f + \sigma_3\,\Delta g)\,dt,$$

$$m_4 = \Delta y_0' + \int_0^t (\rho_4\,\Delta f + \sigma_4\,\Delta g)\,dt.$$

The differentials of x and y are then found by solving the four linear equations (23) for δx, δy, $\delta x'$, $\delta y'$, and these differentials are the first order corrections to x and y required at the time t on the trajectory.

The effectiveness of this method of determining the differential corrections from the standpoint of the computer depends very much upon the plan of the computations and the arrangement of the work. Such a plan has already been described by the writer in the paper last referred to. It turns out that the fundamental system of solutions $(\lambda_i, \mu_i, \rho_i, \sigma_i)$ $(i = 1, \cdots, 4)$, and the cofactors of the determinant of the coefficients in the first members of equations (23), can be determined by three approximate integrations plus elementary arithmetical operations. The system of differential equations actually integrated is the adjoint of the first, second, fourth, and fifth equations of the system (12) which has the form

$$\begin{aligned} A' &= C, \\ B' &= D, \\ C' &= ax'FB - F\left(1 + x'^2\frac{G'}{vG}\right)C - x'y'F\frac{G'}{vG}D, \\ D' &= ay'FB - x'y'F\frac{G'}{vG}C - F\left(1 + y'^2\frac{G'}{vG}\right)D. \end{aligned} \tag{24}$$

If (A, B, C, D) is a solution of this system, and $(\lambda, \mu, \rho, \sigma)$ a solution of the original one, the relation

$$A\lambda + B\mu + C\rho + D\sigma = \text{const.},$$

is always true, and it follows readily that four solutions (A_i, B_i, C_i, D_i) $(i = 1, \cdots, 4)$ of the system (24), with initial values at $t = 0$ forming the identity matrix, will have elements which are the cofactors of the corresponding elements of the determinant of the solutions $(\lambda_i, \mu_i, \rho_i, \sigma_i)$ $(i = 1, \cdots, 4)$

of the original system. The matrix of solutions of the system (24) has the form

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} \tag{25}$$

and it is clear from the form of the equations themselves that when the elements B, C, D of one of the last three rows have been determined by integration of the last three differential equations, the corresponding A can be determined by a quadrature. The elements of the solutions $(\lambda_i, \mu_i, \rho_i, \sigma_i)$ $(i = 1, \cdots, 4)$ are the co-factors of the matrix (25) and they can be calculated by elementary operations of arithmetic.

The differential of z is found by using in equation (7) two solutions (ν, τ) of the third and sixth of equations (12) with initial values $(\nu_1, \tau_1) = (1, 0)$, $(\nu_2, \tau_2) = (0, 1)$ at $t = 0$. These solutions are, respectively,

$$\nu_1 \equiv 1, \qquad \tau_1 \equiv -\frac{x}{x'},$$

$$\nu_2 \equiv 0, \qquad \tau_2 \equiv \frac{x_0'}{x'},$$

as may readily be established by elementary integrations similar to those used in deducing formula (15). Equation (7) then gives

$$\delta z - \frac{x}{x'}\,\delta z' = -\int_0^t \frac{x}{x'}\,\Delta h dt,$$

$$\frac{x_0'}{x'}\,\delta z' = \Delta z_0' + \int_0^t \frac{x_0'}{x'}\,\Delta h dt,$$

which determine the differentials δz and $\delta z'$.

ON THE SUMMABILITY OF THE DEVELOPMENTS IN BESSEL'S FUNCTIONS*

BY

CHARLES N. MOORE

The object of the present paper is to establish sufficient conditions for the summability (Cesàro) at the origin and the uniform summability in the neighborhood of the origin of the developments in Bessel's functions. It has also been the aim of the writer to obtain summability of as low an index as possible without placing any considerable restriction on the function to be developed.

The study of the behavior of the development in the neighborhood of the origin presents much greater difficulties than the study of the development in intervals that do not include the origin. This accounts for the fact that previous discussions of the summability† of the development apply only to intervals of the latter type, and that most of the previous discussions of the convergence of the development are incomplete in the same manner.‡

The difficulty in studying the development in the neighborhood of the origin arises from the fact that the terms in which the asymptotic expansion of the Bessel's functions can be used to advantage begin later and later in the series as we approach the origin. Hence, in studying the series in an interval that reaches up to the origin, the point at which we start using the asymptotic expansion is continually shifting. This gives rise to many complications and accounts for the length of that portion of this paper which

* Presented to the Society, September 4, 1917, and April 12, 1918. Owing to considerations of space some of the results of these papers are reserved for later publication.

† To my knowledge there have been only two such discussions, the first a paper by myself in these Transactions, vol. 10 (1909), pp. 391–435, hereafter referred to as Transactions I, and the second to be found in Ford's *Studies on Divergent Series and Summability*, Chapter V. *Added later.* Since the present paper was written, an article by W. H. Young has appeared (cf. Proceedings of the London Mathematical Society, ser. 2, vol. 18 (1919–20), p. 163), in which both the convergence and the summability are considered, but in this case also the behavior of the development in the neighborhood of the origin is not determined.

‡ As far as I am aware, the only discussion of the convergence of the developments in Bessel's functions which is complete in this respect, is to be found in a paper by myself in these Transactions, vol. 12 (1911), pp. 181–206, to be hereafter referred to as Transactions II.

deals with the uniform summability of the development in the neighborhood of the origin (§§ 7–21).

It may be noted that many of the lemmas obtained, particularly those in sections 7–17, have an interest of their own and many possible applications in other investigations, such as the study of Fourier's series. I have made no attempt to point out explicitly these applications, as I felt that this would detract from the unity of the paper and unnecessarily increase its length.

An asymptotic formula for the roots of a certain equation involving Bessel's functions. §§ 1–2

1. We wish to obtain in this section an asymptotic formula for the positive roots of the equation

$$l\lambda J_\nu'(\lambda) + h J_\nu(\lambda) = 0, \tag{1}$$

where $J_\nu(\lambda)$ represents a Bessel's function of the νth order, ν is a constant positive or zero, and h and l are any constants not both zero. It has already been shown in Transactions I (cf. equation (57)) that the nth positive root of (1) satisfies the equation

$$\lambda_n = n\pi + q + \frac{\psi_1(n)}{n}, \tag{2}$$

where q is a constant dependent on ν and l, and $\psi_1(n)$ *is used to represent any function of* n *that remains finite for all values of* n. The object of the following discussion is to arrive at a more precise expression for the last term.

By means of the asymptotic development for $J_\nu(x)$ and $J_{\nu+1}(x)$, equation (1) may be replaced by the equation*

$$\cos(\lambda - \alpha) + \frac{k}{\lambda}\sin(\lambda - \alpha) + \frac{\psi_1(\lambda)}{\lambda^2} = 0, \tag{3}$$

where k is a constant, and α is a constant that has the value $\frac{1}{4}(2\nu + 3)\pi$ or $\frac{1}{4}(2\nu + 1)\pi$ according as the l of equation (1) is different from or is equal to zero. The corresponding values of q in (2) will be given by†

$$q = r\pi + \frac{\pi}{2} + \alpha \qquad (r \text{ an integer or zero}). \tag{4}$$

We now set

$$\lambda = (n + r)\pi + \frac{\pi}{2} + \alpha + h = m\pi + \frac{\pi}{2} + \alpha + h. \tag{5}$$

In view of (2) and (4) the value of h corresponding to $\lambda = \lambda_n$, which we shall

* Cf. Transactions I, Lemma 3.

† Cf. Transactions I, p. 415. In making use of the equations for q a slight modification is necessary to take account of the fact that the α on page 415 has the same value $\frac{1}{4}(2\nu + 1)$, whether l is or is not zero.

designate by h_n, will be of the form $\psi_1(n)/n$. We shall show further that it is of the form $k/n + \psi_1(n)/n^2$.

From (5) and an application of Maclaurin's expansion we have

$$\cos(\lambda - \alpha) = \cos\left(m\pi + \frac{\pi}{2} + h\right) = (-1)^{m+1}\sin h \tag{6}$$
$$= (-1)^{m+1}\left[h - \frac{h^3}{6}\cos(\theta_1 h)\right],$$

$$\sin(\lambda - \alpha) = \sin\left(m\pi + \frac{\pi}{2} + h\right) = (-1)^m \cos h \tag{7}$$
$$= (-1)^m\left[1 - \frac{h^2}{2}\cos(\theta_2 h)\right],$$

where θ_1 and θ_2 each lie between zero and one. If now we set $h = k/\lambda - c/\lambda^2$ and substitute from (6) and (7) into the left-hand side of (3), it is easy to see that if c and λ are chosen to satisfy the inequalities

$$|\psi_1(\lambda)| + 1 < c < k\lambda, \qquad \lambda > k^3, \tag{8}$$

the left-hand side of (3) will have the sign $(-1)^m$. If we substitute $k/\lambda + c/\lambda^2$ for h, we see that if c and λ satisfy the inequalities (8) the left-hand side of (3) will have the sign $(-1)^{m+1}$. Hence this expression changes sign as h passes from the value $k/\lambda - c/\lambda^2$ to the value $k/\lambda + c/\lambda^2$, and therefore (3) has a root, λ_n, for a value of h, h_n, between these limiting values. Thus we see that h_n is of the form $k/\lambda_n + \psi_1(\lambda_n)/\lambda_n^2$, and consequently, in view of (2), of the form $k/n + \psi_1(n)/n^2$. Therefore, taking account of (4) and (5), we have

$$\lambda_n = n\pi + q + \frac{k}{n} + \frac{\psi_1(n)}{n^2}, \tag{9}$$

the asymptotic formula we desired to obtain.

2. We shall now employ (9) to obtain another formula which will be of use in later reductions. We have from (9)

$$\cos(\lambda_n x - \alpha) = \cos[(n\pi + q)x - \alpha] \tag{10}$$
$$+ \cos\left[(n\pi + q)x - \alpha + \frac{kx}{n} + \frac{x\psi_1(n)}{n^2}\right] - \cos[(n\pi + q)x - \alpha].$$

By means of two elementary trigonometric reductions, the last two terms on the right-hand side of (10) may be put in the form

$$-2\sin[(n\pi + q)x - \alpha]\cos\frac{1}{2}\left(\frac{kx}{n} + \frac{x\psi_1(n)}{n^2}\right)\sin\frac{1}{2}\left(\frac{kx}{n} + \frac{x\psi_1(n)}{n^2}\right)$$
$$-2\cos[(n\pi + q)x - \alpha]\sin^2\frac{1}{2}\left(\frac{kx}{n} + \frac{x\psi_1(n)}{n^2}\right).$$

If we reduce this expression by expanding the sine and cosine of

$$\frac{1}{2}\left(\frac{kx}{n}+\frac{x\,\psi_1(n)}{n^2}\right)$$

by means of the formulas for sin h and cos h given in (6) and (7), and substitute the result for the last two terms of (10), we have finally

$$\cos(\lambda_n x-\alpha)=\cos[(n\pi+q)x-\alpha]-\frac{kx}{n}\sin[(n\pi+q)x-\alpha]+\frac{x\,\psi(x,n)}{n^2}, \tag{11}$$

where $\psi(x,n)$ *is used to represent a function of* x *and* n *which for all fixed values of* n *is continuous in the interval* $(0\leqq x\leqq 1)$ *and which remains finite when* n *increases indefinitely and* x *varies in that interval.*

Summability at the origin of the development of an arbitrary function in terms of Bessel's functions. §§ 3–6

3. We turn now to the consideration of the development

$$\sum_{n=1}^{\infty} A_n J_\nu(\lambda_n x), \tag{12}$$

where

$$A_n=\frac{\int_0^1 x f(x) J_\nu(\lambda_n x)\,dx}{\int_0^1 x[J_\nu(\lambda_n x)]^2\,dx}, \tag{13}$$

and $\lambda_1, \lambda_2, \lambda_3, \cdots$ are the successive positive roots of equation (1). We will first establish some lemmas.

Lemma 1. *If in the interval* $(0\leqq x\leqq 1)$, $f(x)$ *has a Lebesgue integral, is continuous at the origin and such that*

$$\left|\frac{f(x)-f(0)}{x^{\frac{1}{2}}}\right|<M \qquad (0<x\leqq c\leqq 1), \tag{1 }$$

where M *is a positive constant, we have*

$$A_n=\left(M_1\sqrt{n}+\frac{M_2}{\sqrt{n}}\right)\int_0^1\sqrt{x}\,\phi(x)\cos\{(n\pi+q)x-\alpha\}\,dx + \frac{M_3}{\sqrt{n}}\int_0^1\frac{\phi(x)}{\sqrt{x}}(x^2+b)\sin\{(n\pi+q)x-\alpha\}dx + (-1)^n\frac{M_4}{\sqrt{n}}f(0)+\frac{\nu K f(0)}{\lambda_n}+r_n, \tag{15}$$

where A_n is defined by (13), $\phi(x) = f(x) - f(0)$, M_1, M_2, M_3, M_4, q, α, b, *and* K *are constants, and* r_n *is the general term of an absolutely convergent series.*

From the definition of $\phi(x)$ and the substitution $y = \lambda x$, we have

$$(16)\quad \int_0^1 x f(x) J_\nu(\lambda x)\, dx = \int_0^1 x\,\phi(x) J_\nu(\lambda x)\, dx + \frac{f(0)}{\lambda^2}\int_0^\lambda y J_\nu(y)\, dy.$$

But, on integrating by parts,

$$(17)\quad \int_0^\lambda y J_\nu(y)\, dy = \int_0^\infty \frac{1}{y^\nu} y^{\nu+1} J_\nu(y)\, dy = \lambda J_{\nu+1}(\lambda) + \nu \int_0^\lambda J_{\nu+1}(y)\, dy.$$

Moreover,*

$$(18)\quad \int_0^\lambda J_{\nu+1}(y)\, dy = \int_0^\infty J_{\nu+1}(y)\, dy - \int_\lambda^\infty J_{\nu+1}(y)\, dy = K_1 + \frac{\psi_1(\lambda)}{\lambda^{\frac{1}{2}}},$$

where K_1 is a constant.

From the asymptotic expansion† for $J_\nu(\lambda x)$ we have

$$(19)\quad \int_{c/\lambda}^1 x\,\phi(x) J_\nu(\lambda x)\, dx = \frac{A}{\lambda^{\frac{1}{2}}}\int_{c/\lambda}^1 \sqrt{x}\,\phi(x)\cos(\lambda x - \alpha)\, dx$$
$$+ \frac{A'}{\lambda^{\frac{3}{2}}}\int_{c/\lambda}^1 \frac{\phi(x)}{\sqrt{x}}\sin(\lambda x - \alpha)\, dx + \frac{A}{\lambda^{\frac{3}{2}-\rho}}\int_{c/\lambda}^1 \frac{\phi(x)}{x^{\frac{1}{2}}}\cdot\frac{1}{x^{1-\rho}}\cdot\frac{\psi(\lambda x)}{(\lambda x)^\rho}\, dx$$
$$= \frac{A}{\lambda^{\frac{1}{2}}}\int_0^1 \sqrt{x}\,\phi(x)\cos(\lambda x - \alpha)\, dx$$
$$+ \frac{A'}{\lambda^{\frac{3}{2}}}\int_0^1 \frac{\phi(x)}{\sqrt{x}}\sin(\lambda x - \alpha)\, dx + \frac{\psi_1(\lambda)}{\lambda^{\frac{3}{2}-\rho}} \qquad \left(\begin{matrix} A' \text{ a constant} \\ 0 < \rho < \frac{1}{2}\end{matrix}\right),$$

since, in view of (14)

$$\int_0^{c/\lambda} \sqrt{x}\,\phi(x)\cos(\lambda x - \alpha)\, dx = \frac{\psi_1(\lambda)}{\lambda^2},$$

$$\int_0^{c/\lambda} \frac{\phi(x)}{\sqrt{x}}\sin(\lambda x - \alpha)\, dx = \frac{\psi_1(\lambda)}{\lambda}.$$

Furthermore

$$(20)\quad \int_0^{c/\lambda} x\,\phi(x) J_\nu(\lambda x)\, dx = \frac{\psi_1(\lambda)}{\lambda^{\frac{3}{2}}}.$$

Hence finally, from (16), (17), (18), (19), and (20)

* Cf. pp. 187–188 of Transactions II.

† Cf. formula 48 of Transactions I.

$$(21)\qquad \int_0^1 x f(x) J^\nu(\lambda x)\,dx = \frac{A}{\lambda^{\frac{1}{2}}}\int_0^1 \sqrt{x}\,\phi(x)\cos(\lambda x-\alpha)\,dx + \frac{A'}{\lambda^{\frac{3}{2}}}\int_0^1 \frac{\phi(x)}{\sqrt{x}}\sin(\lambda x-\alpha)\,dx + f(0)\frac{J_{\nu+1}(\lambda)}{\lambda} + \frac{\nu K_2 f(0)}{\lambda^2} + \frac{\psi_1(\lambda)}{\lambda^{\frac{5}{2}-\rho}}.$$

Therefore, since*

$$(22)\qquad \int_0^1 x[J^\nu(\lambda x)]^2\,dx = \frac{A^2}{2\lambda} + \frac{A_1\sin 2(\lambda-\alpha)}{\lambda^2} + \frac{\psi_1(\lambda)}{\lambda^3},$$

where A_1 is a constant, we have

$$(23)\qquad A_n = \left(\frac{2\lambda_n^{\frac{1}{2}}}{A} - \frac{4A_1\sin 2(\lambda_n-\alpha)}{A^3\lambda_n^{\frac{1}{2}}}\right)\int_0^1 \sqrt{x}\,\phi(x)\cos(\lambda_n x-\alpha)\,dx + \frac{2A'}{A^2\lambda_n^{\frac{1}{2}}}\int_0^1 \frac{\phi(x)}{\sqrt{x}}\sin(\lambda_n x-\alpha)\,dx + \frac{2f(0)}{A^2}J_{\nu+1}(\lambda_n) + \frac{2\nu K_1 f(0)}{A^2\lambda_n} + \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{3}{2}-\rho}}.$$

But from (2)

$$(24)\qquad \lambda_n^{\frac{1}{2}} = \sqrt{n\pi}\left(1+\frac{q}{n\pi}+\frac{\psi_1(n)}{n^2}\right)^{\frac{1}{2}} = \sqrt{n\pi}\left(1+\frac{q}{2n\pi}+\frac{\psi_1(n)}{n^2}\right).$$

Moreover (Transactions II, equation (42))

$$(25)\qquad \frac{1}{\lambda_n^{\frac{1}{2}}} = \frac{1}{\sqrt{n\pi}} + \frac{\psi_1(n)}{n^{\frac{3}{2}}},$$

and (Transactions I, equation (59))

$$(26)\qquad \sin(\lambda_n x-\alpha) = \sin[(n\pi+q)x-\alpha] + \frac{\psi(x,\lambda_n)}{n}.$$

Similarly we obtain

$$(27)\qquad \sin 2(\lambda_n-\alpha) = \sin 2(n\pi+q-\alpha) + \frac{\psi_1(n)}{n} = \sin 2(q-\alpha) + \frac{\psi_1(n)}{n}.$$

Also, from Transactions II, formula (43),

$$(28)\qquad J_{\nu+1}(\lambda_n) = (-1)^n\frac{C}{\sqrt{n}} + \frac{\psi_1(n)}{n^{\frac{3}{2}}} \qquad (C \text{ a constant}).$$

Combining (24), (25), (11), (26), (27), and (28) with (23), we readily obtain for A_n the form† on the right-hand side of (15). Hence our lemma is proved.

* See Transactions I, p. 403, where a somewhat less explicit form for the left-hand side of (22) is obtained. The present form is derived in similar fashion by using one more term in the asymptotic expansions of $J_\nu(\lambda)$ and $J_{\nu+1}(\lambda)$.

† For the sake of future reference we note that $M_1 = 2/A$, where A is the A of formula 48 (Transactions I).

4. LEMMA 2. *If in the interval* $0 \leqq x \leqq 1$, $\phi(x)$ *has a Lebesgue integral and is furthermore such that*

$$\left|\frac{\phi(x)}{x^{\frac{1}{2}+\rho}}\right| < M \qquad (0 < x \leqq c \leqq 1), \tag{29}$$

where M *is a constant and* ρ *is a constant* > 0, *the series whose general terms are*

$$\begin{aligned} &n\int_0^1 \sqrt{x}\,\phi(x)\cos(qx-\alpha)\cos n\pi x\,dx, \\ &n\int_0^1 \sqrt{x}\,\phi(x)\sin(qx-\alpha)\sin n\pi x\,dx, \end{aligned} \tag{30}$$

where q *and* α *are constants, will be summable* $(C\,1)$.

We will consider first the series whose general term is the first expression in (30). If in that expression we substitute $y = \pi x$, it reduces to

$$\frac{1}{\pi^{\frac{3}{2}}}\int_0^\pi \sqrt{y}\,\phi\left(\frac{y}{\pi}\right)\cos\left(\frac{qy}{\pi}-\alpha\right)n\cos ny\,dy. \tag{31}$$

We shall now obtain the first Cesàro mean for the series we are discussing. We have from a well-known formula*

$$\sum_{m=1}^{m=n}\left(\sum_{r=1}^{r=m}\sin ry\right) = \frac{(n+1)\sin y - \sin(n+1)y}{4\sin^2\frac{1}{2}y}. \tag{32}$$

Differentiating both sides of the above equation with regard to y and making a few trigonometric reductions on the right-hand side we obtain

$$\sum_{m=1}^{m=n}\left(\sum_{r=1}^{r=m} r\cos ry\right) = \frac{\sin(n+\frac{1}{2})y}{4\sin^3\frac{1}{2}y} - \frac{(n+1)+n\cos(n+1)y}{4\sin^2\frac{1}{2}y}. \tag{33}$$

From this equation we obtain at once for the first Cesàro mean of the series whose general term is (31)

$$\begin{aligned} &\frac{1}{4n\pi^{\frac{3}{2}}}\int_0^\pi \frac{\sqrt{y}\,\phi(y/\pi)}{\sin^2\frac{1}{2}y}\cos\left(\frac{qy}{\pi}-\alpha\right)\frac{\sin(n+\frac{1}{2})y}{\sin\frac{1}{2}y}\,dy \\ &\quad -\left(1+\frac{1}{n}\right)\frac{1}{4\pi^{\frac{3}{2}}}\int_0^\pi \frac{\sqrt{y}\,\phi(y/\pi)}{\sin^2\frac{1}{2}y}\cos\left(\frac{qy}{\pi}-\alpha\right)dy \\ &\quad -\frac{1}{4\pi^{\frac{3}{2}}}\int_0^\pi \frac{\sqrt{y}\,\phi(y/\pi)}{\sin^2\frac{1}{2}y}\cos\left(\frac{qy}{\pi}-\alpha\right)\cos(n+1)y\,dy, \end{aligned} \tag{34}$$

where the convergence of each integral, and hence the justification for writing the expression in the form (34), is obtained from the condition (29).

The second term of (34) is readily seen to approach a limit as n becomes infinite. The third term is a constant multiple of a Fourier's coefficient of a

* Cf. Transactions I, p. 413.

function having a Lebesgue integral. Hence from a theorem due to Lebesgue* we know that it approaches zero as a limit as n becomes infinite. It remains then to examine the first term.

It is readily seen that a positive constant, K, exists, such that for all values of y

$$\left|\frac{\sin\,(n+\tfrac{1}{2})\,y}{n\sin\tfrac{1}{2}y}\right| < K. \tag{35}$$

Moreover, a positive ϵ being assigned, we may choose δ such that

$$\frac{1}{4\pi^{\frac{3}{2}}}\int_0^{\delta}\left|\frac{\sqrt{y}\,\phi\,(y/\pi)}{\sin^2\tfrac{1}{2}y}\cos\left(\frac{qy}{\pi}-\alpha\right)\right|dy < \frac{\epsilon}{2K}, \tag{36}$$

and δ being thus chosen, we may then select an m such that

$$\frac{1}{4n\pi^{\frac{3}{2}}}\int_{\delta}^{\pi}\left|\frac{\sqrt{y}\,\phi\,(y/\pi)}{\sin^2\tfrac{1}{2}y}\cos\left(\frac{qy}{\pi}-\alpha\right)\frac{\sin\,(n+\tfrac{1}{2})\,y}{\sin\tfrac{1}{2}y}\right|dy < \frac{\epsilon}{2} \quad (n \geqq m). \tag{37}$$

From (35), (36), and (37) it follows at once that the first term in (34) may be made less than ϵ by choosing $n \geqq m$. Hence this term approaches zero as a limit as n becomes infinite. Thus it follows that (34), or the first Cesàro mean of the series whose general term is the first expression in (30), approaches a limit as n becomes infinite, and the first part of our lemma is proved.

Let us consider now the series whose general term is the second term in (30). If we set $y = \pi x$, this term reduces to

$$\frac{1}{\pi^{\frac{3}{2}}}\int_0^{\pi}\sqrt{y}\,\phi\left(\frac{y}{\pi}\right)\cos\left(\frac{qy}{\pi}-\alpha\right)n\sin ny\,dy.$$

But, from a well-known formula,

$$\sum_{m=1}^{m=n}\left(\sum_{r=1}^{r=m}\cos ry\right) = \frac{\sin^2\tfrac{1}{2}\,(n+1)\,y}{2\sin^2\tfrac{1}{2}y} - \frac{n+1}{2}. \tag{38}$$

Whence, differentiating both sides with respect to y,

$$\sum_{m=1}^{m=n}\left(\sum_{r=1}^{r=m} r\sin ry\right) = \frac{(n+1)\sin\,(n+1)\,y}{4\sin^2\tfrac{1}{2}y} - \frac{\sin^2\tfrac{1}{2}\,(n+1)\,y\cos\tfrac{1}{2}y}{2\sin^3\tfrac{1}{2}y}. \tag{39}$$

Thus we have for the first Cesàro mean of the series we are discussing, the expression

$$\begin{aligned}&\left(1+\frac{1}{n}\right)\frac{1}{4\pi^{\frac{3}{2}}}\int_0^{\pi}\frac{\sqrt{y}\,\phi\,(y/\pi)}{\sin^2\tfrac{1}{2}y}\cos\left(\frac{qy}{\pi}-\alpha\right)\sin\,(n+1)\,y\,dy - \frac{1}{2n\pi^{\frac{3}{2}}}\\ &\times\int^{\pi}\frac{\sqrt{y}\,\phi\,(y/\pi)}{\sin^2\tfrac{1}{2}y}\cos\left(\frac{qy}{\pi}-\alpha\right)\cos\frac{y}{2}\frac{\sin\tfrac{1}{2}\,(n+1)\,y}{\sin\tfrac{1}{2}y}\sin\tfrac{1}{2}(n+1)\,y\,dy.\end{aligned} \tag{40}$$

* Cf. Lebesgue, *Sur les séries trigonométriques*, Annales de l'école normale supérieure, ser. 3, vol. 20 (1903), pp. 471–473.

The integral in the first term is a constant multiple of a Fourier's coefficient of a function having a Lebesgue integral. Hence, by the theorem of Lebesgue quoted above, this term approaches zero as a limit as n becomes infinite. By a discussion entirely analogous to the discussion of the first term of (34) it may be shown that the second term of (40) approaches zero as a limit as n becomes infinite. Hence (40) approaches a limit as n becomes infinite and therefore the series whose general term is the second term of (30) is summable $(C\,1)$. Our lemma is thus completely established.

5. LEMMA 3. *The series whose general terms are*

$$\int_0^1 \frac{\phi(x)}{\sqrt{x}} \cos(qx - \alpha) \sin n\pi x \, dx,$$
$$\int_0^1 \frac{\phi(x)}{\sqrt{x}} \sin(qx - \alpha) \cos n\pi x \, dx, \tag{41}$$

will be summable $(C\,1)$, *provided* $\phi(x)$ *satisfies the conditions of Lemma* 2.

We will consider first the series whose general term is the first expression in (41). If we set $y = \pi x$, this expression reduces to

$$\frac{1}{\pi^{\frac{1}{2}}} \int_0^\pi \frac{\phi(y/\pi)}{\sqrt{y}} \cos\left(\frac{qy}{\pi} - \alpha\right) \sin ny \, dy. \tag{42}$$

Making use of (32) we have for the first Cesàro mean of the series whose general term is (42)

$$\frac{1}{4\sqrt{\pi}}\left(1 + \frac{1}{n}\right) \int_0^\pi \frac{\phi(y/\pi)}{\sqrt{y} \sin \frac{1}{2}y} \cos\left(\frac{qy}{\pi} - \alpha\right) \frac{\sin y}{\sin \frac{1}{2}y} \, dy$$
$$- \frac{1}{4\sqrt{\pi}} \int_0^\pi \frac{\phi(y/\pi)}{\sqrt{y} \sin \frac{1}{2}y} \cdot \frac{\sin(n+1)y}{n \sin \frac{1}{2}y} \cos\left(\frac{qy}{\pi} - \alpha\right) dy, \tag{43}$$

the required property of $\phi(x)$ assuring the convergence of each integral involved. The first term in (43) evidently approaches a limit as n becomes infinite. By a discussion entirely analogous to that for the first term of (34) it may be shown that the second term approaches zero as a limit as n becomes infinite. Hence the whole expression (43) approaches a limit as n becomes infinite, and the series whose general term is the first term of (41) is summable $(C\,1)$.

Consider now the series whose general term is the second expression in (41). By means of the transformation $y = \pi x$ this expression reduces to

$$\frac{1}{\sqrt{\pi}} \int_0^\pi \frac{\phi(y/\pi)}{\sqrt{y}} \sin\left(\frac{qy}{\pi} - \alpha\right) \cos ny \, dy. \tag{44}$$

Making use of (38) we have for the first Cesàro mean of the series whose general term is (44)

$$(45)\qquad \frac{1}{2\sqrt{\pi}}\int_0^{\pi}\frac{\phi(y/\pi)}{\sqrt{y}\sin\frac{1}{2}y}\cdot\frac{\sin\frac{1}{2}(n+1)y}{n\sin\frac{1}{2}y}\sin\tfrac{1}{2}(n+1)y\sin\left(\frac{qy}{\pi}-\alpha\right)dy$$
$$-\frac{1}{2\sqrt{\pi}}\left(1+\frac{1}{n}\right)\int_0^{\pi}\frac{\phi(y/\pi)}{\sqrt{y}}\sin\left(\frac{qy}{\pi}-\alpha\right)dy,$$

where again the convergence of the integrals involved is secured by the condition on $\phi(x)$. The second term of (45) obviously approaches a limit as n becomes infinite, and the first term may be shown to approach zero as a limit as n becomes infinite by a discussion completely analogous to that for the first term of (34). Hence the whole expression (45) approaches a limit as n becomes infinite, and the series whose general term is the second expression in (41) is summable $(C\,1)$.

Our lemma is thus completely established.

6. We are now ready to prove the following theorem:

THEOREM I. *If $f(x)$ is such that $\phi(x)=f(x)-f(0)$, satisfies the conditions of Lemma 2, the development of $f(x)$ in Bessel's functions of order ν $(\nu\geqq 0)$ will be summable $(C\,\frac{1}{2})$ at the origin.*

For the case $\nu>0$ the theorem is trivial, since $J_\nu(0)=0$ for $\nu>0$ and henc all the terms of the series are zero.

For $\nu=0$ the general term of the series (12) becomes A_n since $J_0(0)=1$, and therefore it may be reduced to the expression on the right-hand side of (15). From Lemmas 2 and 3* and a theor m† due to M. Riesz it follows that the first and second terms of this expression are the general terms of series summable $(C\,\frac{1}{2})$. The third and fifth terms are general terms of convergent series and the fourth term reduces to zero. Hence each term is the general term of a series summable $(C\,\frac{1}{2})$ and consequently the whole expression is the general term of such a series. Our theorem is therefore established.

A GENERALIZATION OF THE RIEMANN-LEBESGUE THEOREM AND A FURTHER PRELIMINARY LEMMA. § 7

7. We will next obtain certain sufficient conditions for the uniform summability $(C\,\frac{1}{2})$, in the neighborhood of the origin, of the series (12). We find it necessary to establish a series of lemmas before proceeding to the main theorem.

* In applying Lemma 3, we note that if $\phi(x)$ satisfies the conditions of Lemma 2, so also does the function $\phi(x)(x^2+b)$.

† Cf. Paris Comptes Rendus, 12 June, 1911, also Hardy-Riesz, *The general theory of dirichlet's series.*

LEMMA 4. *If $f(y)$ is integrable (Lebesgue) in the interval (c, d) and $\phi(x, y)$ is a continuous function of x and y in the region $(a \leqq x \leqq b;\ c \leqq y \leqq d)$, then the integrals*

$$\int_c^d f(y)\phi(x, y) \cos ny\, dy, \quad \int_c^d f(y)\phi(x, y) \sin ny\, dy \tag{46}$$

will approach zero uniformly for all values of x in (a, b) as n becomes infinite.

This lemma is a generalization of Lebesgue's theorem that the Fourier's constants of an integrable function approach zero as a limit.* The proof follows the same general lines as the proof of that theorem.

We will consider only the first integral in (46), as the proof for the second integral is entirely analogous. Since $f(y)$ is integrable we can choose M sufficiently great that for the set of points E of (c, d) for which $|f(y)| > M$

$$\left|\int_E f(y)\phi(x, y) \cos ny\, dy\right| < \frac{\epsilon}{4} \qquad \left(\begin{matrix} a \leqq x \leqq b \\ n = 1, 2, 3, \cdots \end{matrix}\right), \tag{47}$$

where ϵ is an arbitrary positive quantity.

Let E_1 be the set of points of (c, d) complementary to E. Since $f(y)$is bounded on E_1, this set may be divided into a finite number of measurable sets $e_1, e_2, \cdots, e_p$, in each of which the oscillation of $f(y)$ is less than

$$\delta = \frac{\epsilon}{4(M + K)(c - d)}, \tag{48}$$

where K is the maximum absolute value of $\phi(x, y)$. Moreover, each set e_q may be enclosed in a finite or infinite set of intervals of measure $m(e_q) + \eta$, where

$$\eta = \frac{\epsilon}{8MKp}. \tag{49}$$

Let I represent the totality of the intervals enclosing the sets e_q and $f_1(y)$ a function of y agreeing with $f(y)$ on E_1, and on the other points of each set of intervals in which an e_q is enclosed defined to have the value of $f(y)$ at some point of e_q. Then

$$\left|\int_{E_1} f(y)\phi(x, y) \cos ny\, dy - \int_I f_1(y)\phi(x, y) \cos ny\, dy\right| < MKp\eta = \frac{\epsilon}{8}. \tag{50}$$

Moreover, since $\phi(x, y)$ is continuous in a closed region and therefore uni-

* L. c. § 4; also Hobson, *Theory of functions of a real variable*, pp. 674–675. Lebesgue's theorem is a generalization to the case of integrals of his type of the corresponding theorem due to Riemann.

formly continuous, we can subdivide the intervals of I in such a way that in each of the new intervals the oscillation of $\phi(x, y)$ is less than δ for any value of x in (a, b). Then in each of these intervals the oscillation of $f_1(y)\phi)x, y)$ is less than $(M + K)\delta$, for we have for any two values of y, y_1 and y_2, in one of these intervals

$$
\begin{aligned}
&|f_1(y_2)\phi(x, y_2) - f_1(y_1)\phi(x, y_1)| \\
&\qquad \leqq |f_1(y_2)| \cdot |\phi(x, y_2) - \phi(x, y_1)| \\
&\qquad\quad + |\phi(x, y_1)| \cdot |f_1(y_2) - f_1(y_1)| < (M + K)\delta.
\end{aligned} \tag{51}
$$

Furthermore, of the subdivided intervals corresponding to each set e_q, we may choose a certain number r_q in descending order of magnitude so that the sum of the remaining intervals is less than η, η being defined by (49). Then we have

$$
\begin{aligned}
&\left| \int_I f_1(y)\phi(x, y) \cos ny\, dy - \sum_{q=1}^{q=p} \sum_{i=1}^{i=r_q} \int_{l_{q,i}} f_1(y)\phi(x, y) \cos ny\, dy \right| \\
&\qquad\qquad < MKp\eta = \frac{\epsilon}{8},
\end{aligned} \tag{52}
$$

where $l_{q,i}$ represents the ith interval of the r_q intervals associated with e_q.

If we represent by $c_{q,i}(x)$ the value of $f_1(y)\phi(x, y)$ for some particular value of y in the interval $l_{q,i}$, we have, in view of (51) and (48),

$$
\begin{aligned}
&\left| \sum_{q=1}^{q=p} \sum_{i=1}^{i=r_q} \left\{ \int_{l_{q,i}} f_1(y)\phi(x, y) \cos ny\, dy - c_{q,i}(x) \int_{l_{q,i}} \cos ny\, dy \right\} \right| \\
&\qquad < (M + K)(c - d)\delta = \frac{\epsilon}{4} \qquad (a \leqq x \leqq b).
\end{aligned} \tag{53}
$$

Also, for all values of x in (a, b),

$$
\left| \sum_{x=1}^{q=p} \sum_{i=1}^{i=r_q} c_{q,i}(x) \int_{l_{q,i}} \cos ny\, dy \right| < \frac{2MK(r_1 + r_2 + \cdots + r_q)}{n}. \tag{54}
$$

If now we choose a value m of n, large enough to make the right-hand side of (54) less than $\frac{1}{4}\epsilon$, we have on combining (47), (50), (52), (53), and (54)

$$
\left| \int_c^l f(y)\phi(x, y) \cos ny\, dy \right| < \epsilon \qquad \left(\begin{matrix} n \geqq m \\ a \leqq x \leqq b \end{matrix} \right).
$$

Our lemma is therefore proved.

LEMMA 5. *If $\phi(x, y)$ is integrable (Lebesgue) with regard to y in the interval $(a \leqq y \leqq b)$ for all values of x such that $(c \leqq x \leqq d)$ and for any positive ϵ*

$$\int_a^c |\phi(x, y)|\, dy < \epsilon \qquad (c \leqq x \leqq d),$$

when c is properly chosen, and if furthermore $F(x, y, n)$ is a function that remains finite for all values of x and y in the region $(a \leqq y \leqq b;\ c \leqq x \leqq d)$ and all values of n, and is such that

$$\lim_{n \to \infty} F(x, y, n) = 0$$

uniformly for all values of x and y in the region $(a + \delta \leqq y \leqq b;\ c \leqq x \leqq d)$, where δ is any quantity > 0, then

$$\lim_{n \to \infty} \int_a^b \phi(x, y) F(x, y, n)\, dy = 0$$

uniformly for all values of x such that $(c \leqq x \leqq d)$.

A positive ϵ being assigned, we may in view of the conditions of the lemma choose δ such that

$$\left| \int_a^{a+\delta} \phi(x, y) F(x, y, n)\, dy \right| < \frac{\epsilon}{2} \qquad (c \leqq x \leqq d). \tag{55}$$

Then, δ being fixed, we can select an m such that

$$\left| \int_{a+\delta}^b \phi(x, y) F(x, y, n)\, dy \right| < \frac{\epsilon}{2} \qquad \left(\begin{matrix} n \geqq m \\ c \leqq x \leqq d \end{matrix} \right). \tag{56}$$

Combining (55) and (56) we obtain

$$\left| \int_a^b \phi(x, y) F(x, y, n)\, dy \right| < \epsilon \qquad \left(\begin{matrix} n \geqq m \\ c \leqq x \leqq d \end{matrix} \right),$$

and our lemma is proved.

Lemmas on the uniform summability of certain types of trigonometric series. §§ 8–12

8. Lemma 6. *If $g(y)$ is integrable (Lebesgue) in the interval $(0 \leqq y \leqq \pi)$, is continuous in the interval $(0 \leqq y \leqq c)$, and is such that $g(y)/y^{1+\rho}$, where $\rho > 0$, remains finite as $y \to 0$, and if furthermore the integral*

$$\int_0^u \frac{g(x + 2t) - 2g(x) + g(x - 2t)}{t^2}\, dt \tag{57}$$

exists and approaches zero uniformly with u for all values of x in the interval $(0 < x \leqq c)$, then the series whose general terms are

$$n \cos nx \int_0^\pi G(y) \cos ny\, dy, \qquad n \sin nx \int_0^\pi G(y) \sin ny\, dy, \tag{58}$$

where

$$G(y) = g(y) - y\frac{g(x)}{x}, \tag{59}$$

will be uniformly summable for all values of x in the interval $(0 < x \leqq c)$.

We will prove the lemma only for the series corresponding to the first term in (58), since the proof for the other case is entirely analogous. This term may be written in the form

$$\frac{1}{2}\int_0^\pi G(y)\, n \cos n(y+x)\, dy + \frac{1}{2}\int_0^\pi G(y)\, n \cos n(y-x)\, dy. \tag{60}$$

In view of (33) the first Cesàro mean for the series whose general term is the first term of (60), is given by

$$\begin{aligned}
&\frac{1}{8}\int_0^\pi \frac{g(y)}{y^2}\cdot\frac{y^2}{\sin^2\frac{1}{2}(y+x)}\cdot\frac{\sin(n+\frac{1}{2})(y+x)}{n\sin\frac{1}{2}(y+x)}\,dy \\
&\quad - \frac{g(x)}{8x\sin^\rho\frac{1}{2}x}\int_0^\pi \frac{\sin^\rho\frac{1}{2}x}{\sin^\rho\frac{1}{2}(y+x)}\cdot\frac{y}{\sin^{2-\rho}\frac{1}{2}(y+x)}\cdot\frac{\sin(n+\frac{1}{2})(y+x)}{n\sin\frac{1}{2}(y+x)}\,dy \\
&\quad - \frac{1}{8}\left(1+\frac{1}{n}\right)\left[\int_0^\pi \frac{g(y)}{y^2}\cdot\frac{y^2}{\sin^2\frac{1}{2}(y+x)}\,dy\right. \\
&\qquad \left. - \frac{g(x)}{x\sin^\rho\frac{1}{2}x}\int_0^\pi \frac{\sin^\rho\frac{1}{2}x}{\sin^\rho\frac{1}{2}(y+x)}\cdot\frac{y}{\sin^{2-\rho}\frac{1}{2}(y+x)}\,dy\right] \\
&\quad - \frac{1}{8}\int_0^\pi \frac{g(y)}{y^2}\cdot\frac{y^2}{\sin^2\frac{1}{2}(y+x)}\cdot\cos(n+1)(y+x)\,dy + \frac{g(x)}{8x\sin^\rho\frac{1}{2}x} \\
&\quad \times \int_0^\pi \frac{\sin^\rho x}{\sin^\rho\frac{1}{2}(y+x)}\cdot\frac{y^{2-\rho}}{\sin^{2-\rho}\frac{1}{2}(y+x)}\cdot\frac{1}{y^{1-\rho}}\cos(n+1)(y+x)\,dy.
\end{aligned} \tag{61}$$

If we break up the integral in the second term of (60) into $\int_0^{2x}$ and $\int_{2x}^\pi$, the first Cesàro mean for the series whose general term is the part of this term corresponding to the second integral may be written in the form (61) provided we replace $(y+x)$ wherever it occurs by $(y-x)$, and use for the lower limits of the integrals $2x$ instead of zero. The discussion of the various terms of this expression is then found to be entirely analogous to the discussion of (61) as written above. We shall therefore carry through the latter discussion and omit the former one.

It follows from Lemma 5 that as n becomes infinite the first and second terms of (61) approach zero uniformly for all values of x in the interval $(0 < x \leqq c)$. That the third term approaches a limit uniformly for all values of x in the same interval as n becomes infinite is readily apparent. If in the fourth and fifth terms we expand $\cos(n+1)(y+x)$ in terms of

functions of $(n+1)y$ and $(n+1)x$, we may show by an application of Lemma 4 that these terms approach zero as a limit uniformly in the interval $(0 < x \leqq c)$ as n becomes infinite. Thus it has been shown that the first term of (60) is the general term of a series that is uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c)$.

We turn now to the discussion of the second term. We pointed out above that the discussion of the portion of this term corresponding to $\int_{2x}^{\pi}$ is analogous to the discussion of the first term. It remains then to consider the portion of the second term corresponding to $\int_0^{2x}$. We have for the first Cesàro mean of the series whose general term is this portion

$$\int_0^{2x} G(y) \frac{\sin\,(n+\tfrac{1}{2})(y-x) - (n+1)\sin\tfrac{1}{2}(y-x) - n\cos\,(n+1)(y-x)\sin\tfrac{1}{2}(y-x)}{8n\sin^3\tfrac{1}{2}(y-x)}\,dy. \tag{62}$$

If in (62) we break up the interval of integration into $(0, x)$ and $(x, 2x)$, set $y - x = -2t$ in the first interval and $y - x = 2t$ in the second, and then recombine and make use of (59), we obtain

$$\begin{aligned} &\frac{1}{4}\int_0^{x/2} \frac{g(x+2t) - 2g(x) + g(x-2t)}{t^2} \cdot \frac{t^2}{\sin^2 t} \cdot \frac{\sin\,(2n+1)t}{n\sin t}\,dt \\ &-\frac{1}{4}\left(1+\frac{1}{n}\right)\int_0^{x/2} \frac{g(x+2t) - 2g(x) + g(x-2t)}{t^2} \cdot \frac{t^2}{\sin^2 t}\,dt \\ &-\frac{1}{4}\int_0^{x/2} \frac{g(x+2t) - 2g(x) + g(x-2t)}{t^2} \cdot \frac{t^2}{\sin^2 t} \cdot \cos 2(n+1)t\,dt, \end{aligned} \tag{63}$$

where the convergence of the various integrals, and hence the justification for breaking up into three terms, follows from the requirement on $g(y)$ that the integral (57) exist. In view of the further requirement that this integral approach zero with u uniformly for all values of x in the interval $(0 < x \leqq c)$ and Lemma 5, it follows that the first term of (63) approaches zero uniformly for all values of x in the same interval as n becomes infinite. That the second term approaches a limit uniformly for values of x in the same interval as n becomes infinite is readily apparent. It remains to discuss the third term.

Given a positive ϵ, we may in view of the condition imposed on the integral (57) choose $\delta < \frac{1}{2}x$ and so small that

$$\left|\frac{1}{4}\int_0^{\delta} \frac{g(x+2t) - 2g(x) + g(x-2t)}{t^2} \cdot \frac{t^2}{\sin^2 t}\cos 2(n+1)t\,dt\right| < \frac{\epsilon}{2}. \tag{64}$$

If in the portion of the third term of (63) corresponding to $\int_{\delta}^{x/2}$ we reverse the transformations previously made, it becomes

$$(65)\qquad -\frac{1}{8}\int_0^{x-2\delta} G(y)\,\frac{1}{\sin^2\frac{1}{2}(y-x)}\cos(n+1)(y-x)\,dy - \frac{1}{8}\int_{x+2\delta}^{2c} G(y)\,\frac{1}{\sin^2\frac{1}{2}(y-x)}\cos(n+1)(y-x)\,dy.$$

If in each term of (65) we expand $\cos(n+1)(y-x)$ in terms of functions of $(n+1)y$ and $(n+1)x$, it follows from Lemma 4 that for large enough values of n (65) is less in absolute value than $\frac{1}{2}\epsilon$ for all values of x in $(0 < x \leqq c)$. Hence the third term of (63) approaches zero uniformly for all values of x in this interval as n becomes infinite.

We have now shown that each term of (63) approaches a limit uniformly for all values of x in the interval $(0 < x \leqq c)$ as n becomes infinite. Hence (63) and therefore (62) has this same property. It follows then that the series whose general term is the portion of the second term of (60) corresponding to $\int_0^{2x}$ is uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c)$. As we have pointed out before the proof of the uniform summability of the series whose general term is the portion corresponding to $\int_{2x}^{\pi}$ is analogous to the proof of the corresponding fact for the first term of (60). Hence each term of (60), and therefore (60) itself or the first term of (58) is the general term of a series that is uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c)$. As pointed out before the proof for the second term of (58) is analogous, and therefore our lemma may be regarded as established.

9. Lemma 7. *If $g(y)$ satisfies the first three conditions of Lemma 6, and furthermore the integral*

$$(66)\qquad \int_0^u \frac{g(x+2t)-g(x-2t)}{t}\,dt$$

exists, and the function

$$(67)\qquad g_1(x,u) = \frac{1}{u}\int_0^u \frac{g(x+2t)-g(x-2t)}{t}\,dt$$

is such that its total variation in the interval $(0 \leqq u \leqq \gamma)$ approaches zero with γ uniformly for all values of x in the interval $(0 < x \leqq c)$, then the series whose general terms are

$$(68)\qquad n\sin nx\int_0^{\pi} G(y)\cos ny\,dy, \qquad n\cos nx\int_0^{\pi} G(y)\sin ny\,dy,$$

where $G(y)$ is defined by (59), will be uniformly summable for all values of x in the interval $(0 < x \leqq c)$.

We will prove the lemma only for the series corresponding to the first term in (68), since the proof for the second term is entirely analogous. The first term may be written in the form

$$(69)\qquad \frac{1}{2}\int_0^\pi G(y)\, n \sin n(y+x)\,dy - \frac{1}{2}\int_0^\pi G(y)\, n \sin n(y-x)\,dy.$$

In view of (39) the first Cesàro mean for the series whose general term is the first term of (69) is given by

$$\begin{aligned}
&\frac{1}{8}\left(1+\frac{1}{n}\right)\int_0^\pi \frac{g(y)}{y^2}\cdot\frac{y^2}{\sin^2\tfrac{1}{2}(y+x)}\cdot\sin(n+1)(y+x)\,dy\\
&-\frac{1}{8}\left(1+\frac{1}{n}\right)\frac{g(x)}{x\sin^\rho\tfrac{1}{2}x}\int_0^\pi \frac{\sin^\rho\tfrac{1}{2}x}{\sin^\rho\tfrac{1}{2}(y+x)}\cdot\frac{y^{2-\rho}}{\sin^{2-\rho}\tfrac{1}{2}(y+x)}\\
&\qquad\qquad\times\frac{1}{y^{1-\rho}}\sin(n+1)(y+x)\,dy\\
(70)\qquad &-\frac{1}{4}\int_0^\pi \frac{g(y)}{y^2}\cdot\frac{y^2}{\sin^2\tfrac{1}{2}(y+x)}\\
&\times\frac{\sin(n+1)\tfrac{1}{2}(y+x)}{n\sin\tfrac{1}{2}(y+x)}\sin(n+1)\tfrac{1}{2}(y+x)\cos\tfrac{1}{2}(y+x)\,dy\\
&+\frac{g(x)}{x\sin^\rho\tfrac{1}{2}x}\int_0^\pi \frac{\sin^\rho\tfrac{1}{2}x}{\sin^\rho\tfrac{1}{2}(y+x)}\cdot\frac{y}{\sin^{2-\rho}\tfrac{1}{2}(y+x)}\\
&\times\frac{\sin(n+1)\tfrac{1}{2}(y+x)}{n\sin\tfrac{1}{2}(y+x)}\sin(n+1)\tfrac{1}{2}(y+x)\cos\tfrac{1}{2}(y+x)dy.
\end{aligned}$$

If we break up the integral in the second term of (69) into $\int_0^{2x}$ and $\int_{2x}^\pi$, the first Cesàro mean for the series whose general term is the part of this term corresponding to the second integral may be written in the form (70) provided we replace $(y+x)$ wherever it occurs by $(y-x)$, and use for the lower limits of the integrals $2x$ instead of zero. The discussion of the various terms of this expression is then found to be entirely analogous to the discussion of (70) as written above. We shall therefore give only the latter discussion.

It follows from Lemma 4 that the first and second terms of (70) approach zero uniformly for all values of x in the interval $(0 < x \leqq c)$ as n becomes infinite. By the use of Lemma 5 we may establish the same property for the third and fourth terms. It follows then that the series whose general term is the first term of (69) is uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c)$.

We turn then to the second term. As pointed out above the discussion of the portion of this term corresponding to $\int_{2x}^\pi$ is analogous to the discussion of the first term. It remains to consider the portion corresponding to $\int_0^{2x}$. We have for the first Cesàro mean of the series whose general term is this portion

$$(71)\quad \frac{1}{8}\int_0^{2x} G(y)\,\frac{(n+1)\sin(n+1)(y-x)\sin\tfrac{1}{2}(y-x) - 2\sin^2(n+1)\tfrac{1}{2}(y-x)\cos\tfrac{1}{2}(y-x)}{n\sin^3\tfrac{1}{2}(y-x)}\,dy.$$

If in (71) we break up the interval of integration into $(0, x)$ and $(x, 2x)$, set $y - x = -2t$ in the first interval and $y - x = 2t$ in the second, and then recombine and make use of (59), we obtain

$$(72)\quad \begin{aligned} &\frac{1}{4}\left(1+\frac{1}{n}\right)\int_0^{x/2}\{g(x+2t)-g(x-2t)\}\frac{\sin 2(n+1)t}{\sin^2 t}\,dt \\ &-\left(1+\frac{1}{n}\right)\frac{g(x)}{x}\int_0^{x/2}\frac{t}{\sin t}\cdot\frac{\sin 2(n+1)t}{\sin t}\,dt \\ &-\frac{1}{2n}\int_0^{\pi/2}\{g(x+2t)-g(x-2t)\}\cos t\,\frac{\sin^2(n+1)t}{\sin^3 t}\,dt \\ &+\frac{2g(x)}{nx}\int_0^{x/2}\frac{t}{\sin t}\cdot\frac{\sin^2(n+1)t}{\sin^2 t}\,t\,dt. \end{aligned}$$

From the fact that $g(x)/x$ approaches zero with x and the well-known properties of Dirichlet's integral and Fejér's integral it follows without difficulty that the second and fourth terms approach a limit uniformly for all values of x in the interval $(0 < x \leqq c)$. It remains to consider the first and third terms.

We may replace the first term by

$$(73)\quad \frac{1}{4}\left(1+\frac{1}{n}\right)\int_0^{x/2}\frac{g(x+2t)-g(x-2t)}{t}\cdot\frac{\sin 2(n+1)t}{t}\,dt,$$

for the difference between the two expressions has the form

$$(74)\quad \frac{1}{4}\left(1+\frac{1}{n}\right)\int_0^{x/2}\{g(x+2t)-g(x-2t)\}\,\frac{t^2-\sin t^2}{t^2\sin^2 t}\sin 2(n+1)t\,dt,$$

which expression, in view of a theorem due to Hobson,* approaches zero uniformly for all values of x in $(0 < x \leqq c)$. That (73) approaches a limit uniformly in the same interval may be shown by a discussion analogous to that given by de la Vallée Poussin in establishing a certain sufficient condition for the convergence of Fourier's series.† The function $g_1(x, u)$ defined by (67) corresponds to the function $F(u)$ introduced in de la Vallée Poussin's discussion, and the modifications necessary to establish uniformity

*Proceedings of the London Mathematical Society, ser. 2, vol. 5 (1907), p. 277; also *Theory of functions of a real variable*, p. 683.

†Cf. *Un nouveau cas de convergence des séries de Fourier*, Palermo Rendiconti, vol. 31 (1911), p. 296.

of approach are readily apparent. It was in order to secure this uniformity that the restriction upon $F(u)$ made by de la Vallée Poussin was replaced in the hypothesis of our lemma by a corresponding restriction of a uniform character.

We turn now to the discussion of the third term of (72). We may replace it by

$$-\frac{1}{2n}\int_0^{x/2}\{g(x+2t)-g(x-2t)\}\frac{\sin^2(n+1)t}{t^3}\,dt, \tag{75}$$

since the difference between the two expressions,

$$-\frac{1}{2}\int_0^{x/2}\{g(x+2t)-g(x-2t)\}\frac{\sin^3 t - t^3\cos t}{t^3\sin^2 t}\cdot\frac{\sin(n+1)t}{n\sin t}\cdot\sin(n+1)t\,dt, \tag{76}$$

is readily seen from Lemma 5 to approach zero uniformly for all values of x in $(0 < x \leqq c)$. It remains to establish the same for (75).

We break up the interval of integration in (75) into $(0, \alpha)$ and $(\alpha, \frac{1}{2}x)$, where α is any positive number $< \frac{1}{2}x$, and consider first the part of (75) corresponding to the former interval. If in this part we integrate by parts, making use of (67), we obtain

$$-\frac{2}{n}g_1(x, 2\alpha)\frac{\sin^2(n+1)\alpha}{\alpha} + 2\left(1+\frac{1}{n}\right)\int_0^{\alpha} g_1(x, 2t)\frac{\sin 2(n+1)t}{t}\,dt - \frac{4}{n}\int_0^{\alpha} g_1(x, 2t)\frac{\sin^2(n+1)t}{t^2}\,dt. \tag{77}$$

That the second term of (77) approaches a limit uniformly for all values of x in $(0 < x \leqq c)$ as $n \to \infty$, provided we properly choose α, follows readily from the condition imposed on $g_1(x, u)$. The proof follows the lines of the well-known treatment of Dirichlet's integral in the theory of the convergence of Fourier's series, the necessary modifications being obvious. The third term of (77) may be replaced by

$$\frac{4}{n}\int_0^{\alpha} g_1(x, 2t)\frac{\sin^2(n+1)t}{\sin^2 t}\,dt, \tag{78}$$

since their difference, for a proper choice of α, is readily seen to approach zero uniformly in $(0 < x \leqq c)$ as n becomes infinite. That (78) approaches a limit uniformly for values of x in the same interval may be shown by a

discussion analogous to Fejér's proof of the summability of Fourier's series,* the modifications to establish uniformity being apparent. The quantity α having been suitably chosen so as to secure the uniform approach to their limits of the second and third terms of (77), we may for this choice of α make the first term of (77) uniformly small for all values of x in $(0 < x \leqq c)$ by choosing n sufficiently large.

Thus we see that each term of (77), and therefore the whole expression, or the part of (75) corresponding to $\int_0^\alpha$, approaches a limit uniformly in $(0 < x \leqq c)$ as $n \to \infty$, provided α is properly chosen. It is then easy to see that for this fixed value of α the part of (75) corresponding to $\int_a^{\pi/2}$ may be made as small as we please by choosing n sufficiently large. Thus it follows that (75), or the third term of (72), approaches a limit uniformly in $(0 < x \leqq c)$ as $n \to \infty$. Since we had previously established this same property for the other three terms of (72), it follows that the whole expression, or (71), has this property. Hence the part of the second term of (69) corresponding to $\int_0^{2r}$ is the general term of a series that is uniformly summable $(C\,1)$ in $(0 < x \leqq c)$. Since the same is true for the part corresponding to $\int_{2r}^{\pi}$ it follows that the whole term has this property. Since we had previously shown the same for the first term, it follows that (69), or the first term of (68) has this same property. The proof for the second term being analogous, our lemma may be regarded as established.

10. LEMMA 8. *If $g(y)$ satisfies the first three conditions of Lemma* 6, *the series whose general terms are*

$$\cos nx \int_0^\pi \frac{G(y)}{y} \cos ny\, dy, \qquad \sin nx \int_0^\pi \frac{G(y)}{y} \sin ny\, dy, \tag{80}$$

where $G(y)$ is defined by (59), *will be uniformly summable $(C\,1)$ for all values of x in the interval $(0 < x \leqq c)$.*

We will prove the lemma only for the series corresponding to the first term in (80), since the proof for the other case is entirely analogous. That term may be written in the form

$$\frac{1}{2}\int_0^\pi \frac{G(y)}{y} \cos n(y+x)\, dy + \frac{1}{2}\int_0^\pi \frac{G(y)}{y} \cos n(y-x)\, dy. \tag{81}$$

In view of (38) the first Cesàro mean for the series whose general term is the first or second term of (81), is given by

*Mathematische Annalen, vol. 58 (1903–04), pp. 54–59. A more compact discussion in which the conditions on the function are in agreement with our present restrictions, may be found in my symposium paper, Bulletin of the American Mathematical Society, ser. 2, vol. 25 (1919), pp. 259–260.

$$\frac{1}{4n}\int_0^\pi \frac{g(y)}{y}\cdot\frac{\sin^2\frac{1}{2}(n+1)(y\pm x)}{\sin^2\frac{1}{2}(y\pm x)}\,dy-\frac{g(x)}{4nx}$$

(82)

$$\times\int_0^\pi \frac{\sin^2\frac{1}{2}(n+1)(y\pm x)}{\sin^2\frac{1}{2}(y\pm x)}\,dy-\frac{1}{4}\left(1+\frac{1}{n}\right)\left\{\int_0^\pi \frac{g(y)}{y}\,dy-\pi\frac{g(x)}{x}\right\},$$

where the upper sign corresponds to the first term, the lower to the second.

It is readily seen that the third term in (82) approaches a limit uniformly for all values of x in the interval $(0<x\leqq c)$ as n becomes infinite. It remains to consider the first and second terms. If in the second term, the upper sign being used, we set $\frac{1}{2}(y+x)=t$, it takes the form

$$\frac{g(x)}{2nx}\int_{x/2}^{\frac{\pi}{2}+\frac{x}{2}}\frac{\sin^2(n+1)t}{\sin^2 t}\,dt. \tag{83}$$

From the second condition imposed on $g(y)$, it is apparent that $g(x)/x$ approaches zero with x. Hence, given an arbitrary positive ϵ, we may choose a corresponding δ such that (83) is less in absolute value than ϵ for all values of x in the interval $(0<x\leqq\delta)$ and all values of n. Then, δ being fixed, we may choose an m such that (83) is less in absolute value than ϵ for all values of x in the interval $(\delta\leqq x\leqq c)$, provided $n\geqq m$. Thus we see that (83), and hence the second term of (82), the upper sign being used, approaches zero uniformly for all values of x in the interval $(0<x\leqq c)$ as n becomes infinite.

We turn now to the consideration of the first term of (82). Since $g(y)/y$ approaches zero with y, we may choose a δ corresponding to an arbitrary positive ϵ, such that

$$\left|\frac{1}{4n}\int_0^\delta \frac{g(y)}{y}\cdot\frac{\sin^2\frac{1}{2}(n+1)(y+x)}{\sin^2\frac{1}{2}(y+x)}\,dy\right|<\frac{\epsilon}{2}\quad\begin{pmatrix}0<x\leqq c\\ n=1,2,3,\cdots\end{pmatrix}. \tag{84}$$

Then, δ being fixed, we may choose m such that

$$\left|\frac{1}{4n}\int_\delta^\pi \frac{g(y)}{y}\cdot\frac{\sin^2\frac{1}{2}(n+1)(y+x)}{\sin^2\frac{1}{2}(y+x)}\,dy\right|<\frac{\epsilon}{2}\quad\begin{pmatrix}0<x\leqq c\\ n\geqq m\end{pmatrix}. \tag{85}$$

Combining (84) and (85), we see that the first term of (82), the upper sign being used, approaches zero as a limit as n becomes infinite, uniformly for all values of x in the interval $(0<x\leqq c)$.

We have now shown that the third term of (82) and the first and second terms, the upper sign being used, each approach a limit uniformly for all values of x in $(0<x\leqq c)$, as n becomes infinite. Hence the first term of (81) is the general term of a series that is uniformly summable $(C\,1)$ in the interval $(0<x\leqq c)$. It remains to establish the same for the second term.

If in the first and second terms of (82), the lower sign being used, we make the transformation $y - x = 2t$, and then combine them into a single term, they take the form

$$(86) \qquad \frac{1}{2n}\int_{-x/2}^{\pi/2-x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x)}{x}\right\}\frac{\sin^2(n+1)t}{\sin^2 t}\,dt.$$

Since $g(x)/x$ is continuous in the closed interval $(0 \leqq x \leqq c)$, and therefore uniformly continuous there, we may choose a δ corresponding to an arbitrary positive ϵ, such that

$$(87) \qquad \left|\frac{1}{2n}\int_{-v}^{u}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x)}{x}\right\}\frac{\sin^2(n+1)t}{\sin^2 t}\,dt\right| < \frac{\epsilon}{2} \qquad \begin{pmatrix} 0 < x \leqq c \\ n = 1, 2, 3, \cdots \\ 0 < v \leqq \delta \\ 0 < u \leqq \delta \end{pmatrix}.$$

Then, δ being fixed, we may choose an m such that

$$(88) \qquad \left|\frac{1}{2n}\int_{-x/2}^{-\delta}\left\{\frac{g(\ +2t)}{x+2t}-\frac{g(x)}{x}\right\}\frac{\sin^2(n+1)t}{\sin^2 t}\,dt\right| < \frac{\epsilon^*}{4}$$

$$(89) \qquad \left|\frac{1}{2n}\int_{\delta}^{\pi-x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x)}{x}\right\}\frac{\sin^2(n+1)t}{\sin^2 t}\,dt\right| < \frac{\epsilon}{4} \qquad \begin{pmatrix} 0 < x \leqq c \\ n \geqq m \end{pmatrix}.$$

Combining (87), (88), and (89), we see that the first and second terms of (82), the lower sign being used, approach zero as a limit uniformly in $(0 < x \leqq c)$ as n becomes infinite. Since the third term also approaches a limit uniformly in this same interval, it follows that the second term of (81) is the general term of a series that is uniformly summable $(C\,1)$ in that interval. As we have previously shown that the first term is the general term of such a series, it follows that the whole expression (81) or the first term in (80) has the same property. Hence our lemma is proved for this term and, as pointed out before, the proof for the second term is analogous.

11. Lemma 9. *If $g(y)$ satisfies the conditions of Lemma 7, the series whose general terms are*

$$(90) \qquad \cos nx\int_0^{\pi}\frac{G(y)}{y}\sin ny\,dy, \qquad \sin nx\int_0^{\pi}\frac{G(y)}{y}\cos ny\,dy,$$

where $G(y)$ is defined by (59), *will be uniformly summable $(C\,1)$ for all values of x in the interval $(0 < x \leqq c)$.*

We will consider only the first term in (90), the proof for the second term being analogous. The first term may be written in the form

$$(91) \qquad \frac{1}{2}\int_0^{\pi}\frac{G(y)}{y}\sin n(y+x)\,dy + \frac{1}{2}\int_0^{\pi}\frac{G(y)}{y}\sin n(y-x)\,dy.$$

In view of (32) the first Cesàro mean for the series whose general term is the

* If $\frac{1}{2}x < \delta$, (88) is a consequence of (87) for all values of n; if $\frac{1}{2}x \geqq \delta$ we choose m so as to make both (88) and (89) hold.

first term of (91), is given by

$$\begin{aligned}
&\frac{1}{4}\left(1+\frac{1}{n}\right)\int_0^\pi \frac{g(y)}{y^2}\cdot\frac{y}{\sin\frac{1}{2}(y+x)}\cdot\cos\tfrac{1}{2}(y+x)\,dy\\
&\quad-\frac{1}{4}\left(1+\frac{1}{n}\right)\frac{g(x)}{x^{1+\rho}}\int_0^\pi \frac{x^\rho}{(y+x)^\rho}\cdot\frac{y^{1-\rho}}{(y+x)^{1-\rho}}\\
&\qquad\qquad\times\frac{y+x}{\sin\frac{1}{2}(y+x)}\cdot\cos\tfrac{1}{2}(y+x)\cdot\frac{1}{y^{1-\rho}}\,dy\\
&\quad-\frac{1}{8}\int_0^\pi \frac{g(y)}{y^2}\cdot\frac{y}{\sin\frac{1}{2}(y+x)}\cdot\frac{\sin(n+1)(y+x)}{n\sin\frac{1}{2}(y+x)}\,dy\\
&\quad+\frac{1}{8}\frac{g(x)}{x^{1+\rho}}\int_0^\pi \frac{x^\rho}{(y+x)^\rho}\cdot\frac{y^{1-\rho}}{(y+x)^{1-\rho}}\\
&\qquad\qquad\times\frac{y+x}{\sin\frac{1}{2}(y+x)}\cdot\frac{\sin(n+1)(y+x)}{n\sin\frac{1}{2}(y+x)}\cdot\frac{1}{y^{1-\rho}}\,dy,
\end{aligned}\tag{92}$$

where the ρ is the ρ of the third condition in Lemma 6. From the conditions imposed on $g(y)$ it follows readily that each term of (92) approaches a limit uniformly for all values of x in $(0 < x \leqq c)$, as n becomes infinite. Hence the first term of (91) is the general term of a series that is uniformly summable $(C\,1)$ in the above interval. It remains to establish the same for the second term.

The first Cesàro mean for the series whose general term is the second term, may be written in the form

$$\begin{aligned}
&\frac{1}{4}\left(1+\frac{1}{n}\right)\int_0^\pi\left\{\frac{g(y)}{y}-\frac{g(x)}{x}\right\}\cot\tfrac{1}{2}(y-x)\,dy\\
&\quad-\frac{1}{8n}\int_0^\pi\left\{\frac{g(y)}{y}-\frac{g(x)}{x}\right\}\frac{\sin(n+1)(y-x)}{\sin^2\frac{1}{2}(y-x)}\,dy.
\end{aligned}\tag{93}$$

If we break up the interval of integration for each integral in (93) into two parts $(0, 2x)$ and $(2x, \pi)$, the part of (93) which corresponds to the second interval may be put in a form that differs from (92) only by having $2x$ for the lower limit of the integrals, instead of zero, and by having $(y - x)$ wherever $(y + x)$ occurs in (92). This part of (93) is readily seen to approach a limit uniformly in $(0 < x \leqq c)$ as n becomes infinite. It remains to consider the other part.

If in this latter part we set $y - x = 2t$, it may be written in the form,

$$\begin{aligned}
&\frac{1}{2}\left(1+\frac{1}{n}\right)\int_{-\frac{x}{2}}^{\pi/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x)}{x}\right\}\cot t\,dt\\
&\quad-\frac{1}{4n}\int_{-\frac{x}{2}}^{\pi/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x)}{x}\right\}\frac{\sin 2(n+1)t}{\sin^2 t}\,dt.
\end{aligned}\tag{94}$$

If we break up the intervals of integration of the integrals in (94) into $(-\frac{1}{2}x, 0)$ and $(0, \frac{1}{2}x)$, set $t = -t$ in the integrals over the first interval and then recombine, we obtain

$$\frac{1}{2}\left(1+\frac{1}{n}\right)\int_0^{x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x-2t)}{x-2t}\right\}\cot t\,dt - \frac{1}{4n}\int_0^{x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x-2t)}{x-2t}\right\}\frac{\sin 2(n+1)t}{\sin^2 t}dt. \tag{95}$$

The first term of (95) may be written in the form

$$\frac{1}{2}\left(1+\frac{1}{n}\right)\int_0^{x/2}\frac{1}{x+2t}\{g(x+2t)-g(x-2t)\}\cot t\,dt - 2\left(1+\frac{1}{n}\right)\frac{1}{x}\int_0^{x/2}\frac{x}{x+2t}\cdot\frac{g(x-2t)}{x-2t}\cdot t\cot t\,dt. \tag{96}$$

It is readily seen that the second term of (96) approaches a limit uniformly for all values of x in $(0 < x \leqq c)$ as n becomes infinite. It remains to establish the same for the first term. That term may be written in the form

$$\frac{n+1}{2n}\cdot\frac{1}{x}\int_0^{x/2}\frac{g(x+2t)-g(x-2t)}{t}dt + \frac{n+1}{2n}\int_0^{x/2}\{g(x+2t)-g(x-2t)\}\frac{t\cos t-\sin t}{(x+2t)t\sin t}dt - \frac{n+1}{n}\cdot\frac{1}{x}\int_0^{x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x-2t)}{x+2t}\right\}dt. \tag{97}$$

It follows from the conditions imposed upon $g(y)$ that each term of (97) approaches a limit uniformly in $(0 < x \leqq c)$ as n becomes infinite. Hence (96), and therefore the first term of (95), has this same property. We have now to prove the same for the second term of (95).

This term may be written in the form

$$-\frac{1}{4n}\int_0^{x/2}\frac{1}{x+2t}\{g(x+2t)-g(x-2t)\}\frac{\sin 2(n+1)t}{\sin^2 t}dt + \frac{1}{x}\int_0^{x/2}\frac{x}{x+2t}\cdot\frac{g(x-2t)}{x-2t}\cdot\frac{t}{\sin t}\cdot\frac{\sin 2(n+1)t}{n\sin t}dt. \tag{98}$$

The second term of (98) may readily be shown to approach zero uniformly in $(0 < x \leqq c)$ as n becomes infinite. It remains to consider the first term. This term may be put in the form

$$-\frac{1}{4x}\int_0^{x/2}\frac{g(x+2t)-g(x-2t)}{t}\cdot\frac{\sin 2(n+1)t}{n\sin t}\,dt$$

$$(99)\qquad -\frac{1}{4}\int_0^{x/2}\{g(x+2t)-g(x-2t)\}\frac{t-\sin t}{(x+2t)t\sin t}\cdot\frac{\sin 2(n+1)}{n\sin t}\,dt$$

$$+\frac{1}{2x}\int_0^{x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x-2t)}{x+2t}\right\}\frac{\sin 2(n+1)t}{n\sin t}\,dt.$$

The second and third terms of (99) are readily seen to approach zero as a limit uniformly in $(0<x\leqq c)$ as n becomes infinite. The first term requires further consideration.

If in this term we integrate by parts, making use of (67), we obtain

$$-\frac{1}{4nx}\left[4tg_1(x,2t)\frac{\sin 2(n+1)t}{\sin t}\right]_0^{x/2}$$

$$+\frac{1}{4nx}\int_0^{x/2}4tg_1(x,2t)\left[\frac{2(n+1)\cos 2(n+1)t}{\sin t}\right.$$

$$\left.-\frac{\sin 2(n+1)t\cos t}{\sin^2 t}\right]dt$$

$$(100)\qquad =-\frac{g_1(x,x)\sin(n+1)x}{2n\sin\frac{1}{2}x}$$

$$+2\left(1+\frac{1}{n}\right)\frac{1}{x}\int_0^{x/2}g_1(x,2t)\frac{t}{\sin t}\cos 2(n+1)t\,dt$$

$$-\frac{1}{nx}\int_0^{x/2}g_1(x,2t)\frac{t\cos t}{\sin t}\cdot\frac{\sin 2(n+1)t}{\sin t}\,dt.$$

We are going to show that the right-hand side of (100) approaches zero uniformly for all values of x in the interval $(0\leqq x<c)$ as n becomes infinite. We must first establish an additional property of $g_1(x,u)$.

If we extend the region of definition of $g(y)$ to the left of the origin by setting $g(-y)=-g(y)$, it is apparent from the third condition imposed on $g(y)$ in Lemma 6 that it possesses a zero derivative at the origin. Hence $g_1(0,u)$ approaches zero with u, and since $g_1(x,u)$ is a continuous function of x for a fixed $u>0$ and approaches a limit uniformly for all values of x in $(0<x\leqq c)$ as u approaches zero, it follows that we may make $g_1(x,u)$ as small as we please if we choose x and u sufficiently small. Therefore we may choose a positive δ corresponding to an arbitrary positive ϵ, such that the right-hand side of (100) is less in absolute value than ϵ for all values of x in $(0<x\leqq\delta)$ and all values of n. Then being fixed, we may choose m so large that this same expression is less in absolute value than ϵ for $(\delta<x$

$\leqq c$) and values of $n \geqq m$.* Hence it follows that the right-hand side of (100), and therefore the first term of (99), approaches zero uniformly in $(0 < x \leqq c)$ as n becomes infinite. Therefore (99) and hence (98), or the second term of (95) has this same property.

We have now shown that each term of (95) approaches a limit uniformly in $(0 < x \leqq c)$ as n becomes infinite. Hence (95), and therefore (94), has this property. Since (94) is part of (93) and the other part has this property, it follows that (93) has it. Therefore the second term of (91) is the general term of a series that is uniformly summable $(C\,1)$ in $(0 < x \leqq c)$. Since we previously showed that the first term was the general term of such a series, it follows that (91), and therefore the first term in (90), has this same property. Hence our lemma is proved for the first term of (90) and, as pointed out before, the proof for the second term is analogous.

12. LEMMA 10. *The series whose general terms are*

$$(-1)^n \cos n\pi x, \qquad (-1)^n \sin n\pi x, \tag{101}$$

are uniformly summable $(C\,1)$ *in the interval* $(0 \leqq x \leqq c < 1)$.

We may find the sum of the first n terms of each of these two series by first finding the sum of the first n terms of the series whose general term is $(-1)^n e^{in\pi x}$ and then taking the real and imaginary parts respectively for the two sums we wish to obtain. We thus find for the two sums in question the expressions

$$\begin{gathered} -\frac{1}{2} + (-1)^n \frac{\cos n\pi x + \cos (n+1)\pi x}{2(1+\cos \pi x)}, \\ \frac{-\sin \pi x + (-1)^n \{\sin n\pi x + \sin (n+1)\pi x\}}{2(1+\cos \pi x)}. \end{gathered} \tag{102}$$

From these expressions we readily obtain for the arithmetic mean of the first n sums in each of the two cases the following results

$$\begin{gathered} -\frac{1}{2} - \frac{\cos \pi x + (-1)^{n+1} \cos (n+1)\pi x}{2n(1+\cos \pi x)}, \\ -\frac{\sin \pi x}{2(1+\cos \pi x)} - \frac{\sin \pi x + (-1)^{n+1} \sin (n+1)\pi x}{2n(1+\cos \pi x)}. \end{gathered} \tag{103}$$

It is easy to see that each of the two expressions in (103) approaches a limit uniformly in the interval $(0 \leqq x \leqq c < 1)$. Our lemma is therefore proved.

LEMMA 11. *The series whose general terms are*

$$n \cos nx \int_0^\pi y \sin ny\, dy, \qquad n \sin nx \int_0^\pi y \sin ny\, dy \tag{104}$$

are uniformly summable in the interval $(0 \leqq x \leqq c < \pi)$.

* In connection with the second term on the right-hand side of (100) we need to apply here Lemma 4.

By means of the transformation $ny = z$, the integrals in (104) may be reduced to a form that is readily integrated, whence it is seen that the two expressions in (104) reduce to

$$\pi[(-1)^{n+1}\cos nx], \qquad \pi[(-1)^{n+1}\sin nx].$$

It follows from Lemma 10 and a change of variable that each of these two expressions is the general term of a series that is uniformly summable in the interval $(0 \leqq x \leqq c < \pi)$. Hence the expressions in (104) have the same property, and our lemma is proved.

LEMMAS ON THE UNIFORM CONVERGENCE OF CERTAIN TYPES OF TRIGONOMETRIC SERIES. §§ 13-14

13. LEMMA 12. *The series whose general terms are*

$$n\cos nx\int_0^\pi y\cos ny\,dy, \qquad n\sin nx\int_0^\pi y\cos ny\,dy \tag{108}$$

are uniformly convergent in the interval $(0 \leqq x \leqq c < \pi)$.

By means of the transformation used in Lemma 11 we are able to evaluate the integral in each of the terms of (108). These terms reduce thus to the form

$$(-1)^n\frac{\cos nx}{n}, \qquad (-1)^n\frac{\sin nx}{n}. \tag{109}$$

We have shown in Lemma 10 that the series whose general terms are the expressions in (101) are uniformly summable $(C\,1)$ in the interval $(0 \leqq x \leqq c < 1)$. From a combination of this fact, a change of variable, and the theorem of M. Riesz referred to in § 6, we are able to infer at once the truth of our lemma.

14. LEMMA 13. *The series whose general term is*

$$x^\rho\cos nx\int_0^\pi \sin ny\,dy, \tag{110}$$

where ρ *is any positive constant, is uniformly convergent in the interval* $(0 \leqq x \leqq c < \pi)$.

On evaluating the integral in (110), this expression reduces to the form $(-2x^\rho\cos nx)/n$ when n is odd, and to zero when n is even. Hence the proof of our lemma resolves itself into establishing the uniform convergence of the series

$$x^\rho\cos x + \frac{x^\rho\cos 3x}{3} + \frac{x^\rho\cos 5x}{5} + \cdots \tag{111}$$

in the interval in question.

We consider first the series

$$\sin x + \sin 3x + \sin 5x + \cdots . \tag{112}$$

By the device of summing the first n terms of the geometric series $e^{ix} + e^{3ix} + e^{5ix} + \cdots$ and taking the pure imaginary part of that sum we obtain for the sum of the first n terms of (112), provided x is such that $\sin x \neq 0$,

$$\frac{1}{2} - \frac{\cos 2nx}{2 \sin x}. \tag{113}$$

But since

$$\int_x^{\pi/2} \sin nx dx = \frac{\cos nx}{n} \qquad (n = 1, 3, 5, \cdots), \tag{114}$$

we hav

$$\sum_{m=0}^{m=n-1} x^\rho \frac{\cos (2m+1)x}{2m+1} = x^\rho \left(\frac{\pi}{4} - \frac{x}{2} \right) - x^\rho \int_x^{\pi/2} \frac{\cos 2nx}{2 \sin x} dx. \tag{115}$$

It is apparent from the theorem of Lebesgue referred to in § 4 that the second term on the right-hand side of (115) approaches zero for any value of $x > 0$. If we can show that for any given positive ϵ we can choose n so large that this term is less in absolute value than ϵ for all values of x in $(0 < x \leqq c < \pi)$, our lemma will be established.*

We have

$$\left| x^\rho \int_x^{\pi/2} \frac{\cos 2nx}{2 \sin x} dx \right| < x^\rho \int_x^{\pi/2} \frac{dx}{x} = x^\rho \left[\log \frac{\pi}{2} - \log x \right]. \tag{116}$$

We may therefore choose a δ such that the second term of (115) is less in absolute value than ϵ when $(0 < x \leqq \delta)$, for all values of n. Then, δ being fixed, we may choose an m so large that this same term is less in absolute value than ϵ when $(\delta < x \leqq c < \pi)$ and $n \geqq m$.† Hence, as pointed out before, our lemma is established.

LEMMA 14. *The series whose general term is*

$$x^\rho \sin nx \int_0^\pi \sin ny \, dy, \tag{117}$$

where ρ is any positive constant, is uniformly convergent in the interval $(0 \leqq x \leqq c < \pi)$.

* The convergence of the series for $x = 0$ is obvious, since each term is equal to zero in that case.

† This follows from the relationship

$$\int_x^{\pi/2} \frac{\cos 2nx}{\sin x} dx = -\frac{\sin 2nx}{4n \sin x} + \frac{1}{4n} \int_x^{\pi/2} \frac{\sin 2nx}{\sin^2 x} \cos x \, dx,$$

obtained from an integration by parts.

If we evaluate the integral in (117), we find that the proof of our lemma reduces to proving the uniform convergence of the series

$$x^\rho \sin x + \frac{x^\rho \sin 3x}{3} + \frac{x^\rho \sin 5x}{5} + \cdots \tag{118}$$

in the interval in question.

We consider first the series

$$\cos x + \cos 3x + \cos 5x + \cdots . \tag{119}$$

By the device of the previous lemma we obtain for the sum of the first n terms of (119)

$$\frac{\sin 2nx}{2 \sin x} . \tag{120}$$

But since

$$\int_x^{\pi/2} \cos nx \, dx = \left[\frac{\sin nx}{n}\right]_x^{\pi/2} = \frac{(-1)^{(n-1)/2}}{n} - \frac{\sin nx}{n} \tag{121}$$

$$(n = 1, 3, 5, \cdots),$$

we have

$$\sum_{m=0}^{m=n-1} \left[x^\rho \frac{(-1)^m}{2m+1} - x^\rho \frac{\sin (2m+1)x}{2m+1} \right] = x^\rho \int_x^{\pi/2} \frac{\sin 2nx}{2 \sin x} dx . \tag{122}$$

Equation (122) furnishes a fresh proof of the well-known fact* that the series (118) without the factor x^ρ, converges to a constant value for all values of x between 0 and π. We shall use it further to prove the convergence of (118) to be uniform in the interval $(0 \leqq x \leqq c < \pi)$.

We have from (122)

$$\left| \sum_{m=0}^{m=n-1} x^\rho \frac{(-1)^m}{2m+1} - \sum_{m=0}^{m=n-1} x^\rho \frac{\sin (2m+1)x}{2m+1} \right| < x^\rho \int_x^{\pi/2} \frac{1}{x} dx \tag{123}$$

$$= x^\rho \left[\log \frac{\pi}{2} - \log x \right].$$

Given an arbitrary positive ϵ, we may choose δ so that the right-hand side of (123) is less in absolute value than $\frac{1}{2}\epsilon$ for all values of x such that $(0 < x \leqq \delta)$. We may then choose an m_1 so that for the same values of x and all values of $n \geqq m_1$ the first summation on the left-hand side of (123) differs from the value to which it converges, $\frac{1}{4}\pi x^\rho$, by less than $\frac{1}{2}\epsilon$. Then we have

$$\left| \tfrac{1}{4}\pi x^\rho - \sum_{m=0}^{m=n-1} x^\rho \frac{\sin (2m+1)x}{2m+1} \right| < \epsilon \qquad \begin{pmatrix} 0 < x \leqq \delta \\ n \geqq m_1 \end{pmatrix}. \tag{124}$$

Now, δ being fixed, we may choose an m_2 so large that the right-hand side of (122) is less in absolute value than $\frac{1}{2}\epsilon$ for all values of x in $(\delta \leqq x \leqq c)$ and

* Cf. Byerly, *Fourier's Series*, etc., p. 39.

all values of $n \geqq m_2$,* while the summation corresponding to the first term on the left-hand side of (122) differs from $\frac{1}{4}\pi x^\rho$ by a quantity less in absolute value than $\frac{1}{2}\epsilon$. Then we have

$$\left| \tfrac{1}{4}\pi x^\rho - \sum_{m=0}^{m=n-1} x^\rho \frac{\sin(2m+1)x}{2m+1} \right| < \epsilon \qquad \begin{pmatrix} \delta \leqq x \leqq c \\ n \geqq m_2 \end{pmatrix}. \tag{125}$$

Taking m as the larger of the two numbers m_1 and m_2, we have from a combination of (124) and (125)

$$\left| \tfrac{1}{4}\pi x^\rho - \sum_{m=0}^{m=n-1} x^\rho \frac{\sin(2m+1)x}{2m+1} \right| < \epsilon \qquad \begin{pmatrix} 0 < x \leqq c < \pi \\ n \geqq m \end{pmatrix}. \tag{126}$$

Our lemma is therefore proved since the convergence of the series for $x = 0$ is obvious.

Further lemmas on the uniform summability of certain types of series. §§ 15–17

15. Lemma 15. *If $\phi(y)$ satisfies the conditions of Lemma 2 and if furthermore $\chi(y) = \sqrt{y}\,\phi(y)$ satisfies the fourth condition imposed on $g(y)$ in Lemma 6 and the second and third conditions imposed on $g(y)$ in Lemma 7, then the functions*

$$\begin{aligned} &\sqrt{y}\,\phi(y)\cos(qy-\alpha), \qquad &&\sqrt{y}\,\phi(y)\sin(qy-\alpha), \\ &\sqrt{y}\,\phi(y)y^2\cos(qy-\alpha), \qquad &&\sqrt{y}\,\phi(y)y^2\sin(qy-\alpha), \end{aligned} \tag{130}$$

will satisfy the conditions imposed on $g(y)$ in Lemmas 6 and 7.

We will consider only the first function of (130) since the proof for the other functions is analogous. From the fact that $\phi(y)$ satisfies the conditions of Lemma 2, it is readily apparent that the first function of (130) satisfies the first three conditions of Lemma 6 and therefore the first condition of Lemma 7. If we represent this function by $F(y)$, we have

$$\begin{aligned} F(y+2t) - 2F(y) + F(y-2t) &= \cos(qy-\alpha)[\chi(y+2t) - 2\chi(y) + \chi(y-2t)] \\ &\quad - 2\sin qt \sin(qy+qt-\alpha)[\chi(y+2t) - \chi(y-2t)] \\ &\quad - 4\sin^2 qt \cos(qy-\alpha)\chi(y-2t). \end{aligned} \tag{131}$$

$$\begin{aligned} F(y+2t) - F(y-2t) &= \cos(qy-\alpha)[\chi(y+2t) - \chi(y-2t)] \\ &\quad - 2\sin qt \sin(qy-qt-\alpha)\chi(y-2t) \\ &\quad - 2\sin qt \sin(qy+qt-\alpha)\chi(y+2t). \end{aligned} \tag{132}$$

From these two equations and the conditions imposed on $\chi(y)$ it is readily

* That this is possible follows from an integration by parts in the integral on the right-hand side of (122) similar to that given in a previous footnote for the integral on the right-hand side of (115).

inferred that $F(y)$, or the first function in (130), satisfies the fourth condition imposed on $g(y)$ in Lemma 6 and the second and third conditions imposed on $g(y)$ in Lemma 7. Hence the present lemma is established.

We find it convenient to introduce here the following notation,

$$\Phi_n(x) = A\cos[(n\pi+q)x-\alpha] + B[\sin(n\pi+q)x-\alpha]. \tag{133}$$

We are now ready for the proof of the following lemma:

LEMMA 16. *The series whose general terms are*

$$\begin{aligned}
&n\Phi_n(x)\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos n\pi y\,dy,\\
&n\Phi_n(x)\int_0^1 \sqrt{y}\,\phi(y)\sin(qy-\alpha)\sin n\pi y\,dy,\\
&\Phi_n(x)\int_0^1 \frac{\phi(y)}{\sqrt{y}}(y^2+b)\sin(qy-\alpha)\cos n\pi y\,dy,\\
&\Phi_n(x)\int_0^1 \frac{\phi(y)}{\sqrt{y}}(y^2+b)\cos(qy-\alpha)\sin n\pi y\,dy,
\end{aligned} \tag{134}$$

will be uniformly summable $(C\,1)$ *in the interval* $(0<x\leqq c<1)$, *provided* $\phi(y)$ *satisfies the conditions of Lemma* 15.

We will carry through the proof only for the first term in (134), since the proof for the other terms is analogous. If we substitute in this term for $\Phi_n(x)$ its value, it takes the form

$$\begin{aligned}
&\{A\cos(qx-\alpha)+B\sin(qx-\alpha)\}n\cos n\pi x\\
&\qquad\times\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos n\pi y\,dy\\
&+\{B\cos(qx-\alpha)-A\sin(qx-\alpha)\}n\sin n\pi x\\
&\qquad\times\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos n\pi y\,dy.
\end{aligned} \tag{135}$$

Since $\phi(y)$ satisfies the conditions of Lemma 15, it follows from that lemma that $\sqrt{y}\,\phi(y)\cos(qy-\alpha)$ satisfies the conditions of Lemmas 6 and 7. Hence from these lemmas and a change of variable we infer that the series whose general terms are

$$\begin{aligned}
&n\cos n\pi x\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos n\pi y\,dy\\
&\qquad-\frac{\phi(x)\cos(qx-\alpha)}{\sqrt{x}}\,n\cos n\pi x\int_0^1 y\cos n\pi y\,dy,\\
&n\sin n\pi x\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos n\pi y\,dy\\
&\qquad-\frac{\phi(x)\cos(qx-\alpha)}{\sqrt{x}}\,n\sin n\pi x\int_0^1 y\cos n\pi y\,dy,
\end{aligned} \tag{136}$$

are uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c < 1)$. From Lemma 12 and a change of variable we infer the same property for the series whose general terms are the second terms of the expressions in (136). Hence it follows that the series whose general terms are the first terms in (136) have this property and therefore that the series whose general term is (135) has the same property. Thus our lemma is proved.

16. LEMMA 17. *If the series* $\sum u_n(y)$ *is uniformly summable* $(C\,1)$ *or uniformly convergent in the interval* $(b < y \leqq d)$, *then the series* $\sum u_n(y) f_n(x)$, *where*

$$f_n(x) = 1 \qquad (n = 1, 2, \cdots, m+1);$$

$$f_n(x) = \frac{a^{\frac{1}{2}}}{n^{\frac{1}{2}} x^{\frac{1}{2}}} \qquad (n = m+2, m+3, \cdots), \tag{137}$$

a being a positive constant and m being used to represent the largest integer such that $(0 < mx < a)$, *will be uniformly summable* $(C\,1)$ *or uniformly convergent respectively, in the region* $(b < y \leqq d;\ 0 < x \leqq c)$, *where c is any positive constant.*

For the case of convergence the proof follows readily from Abel's lemma.

For the case of summability the lemma may be proved by showing that the functions (137) satisfy, throughout the interval $(0 < x \leqq c)$ the conditions of a theorem* due to G. H. Hardy. These conditions require that a positive constant K exist such that

$$\sum_{n=\mu}^{n=\nu} (n+1)\,|\,\Delta^2 f_n(x)\,| < K, \qquad \sum_{n=\mu}^{n=\nu} |\,\Delta f_n\,| < K, \tag{138}$$

for all values of μ and ν and all values of x in $(0 < x \leqq c)$. It will only be necessary to establish the first inequality in (138), since by virtue of a lemma† due to Bromwich the second is a consequence of the first.

It is readily seen that the first inequality is equivalent to

$$\sum_{n=1}^{\infty} (n+1)\,|\,\Delta^2 f_n(x)\,| < K \qquad (0 < x \leqq c). \tag{139}$$

Moreover,

$$\begin{aligned}\sum_{n=1}^{\infty} (n+1)\,|\,\Delta^2 f_n(x)\,| &= (m+1)\left(1 - \frac{a^{1/2}}{(m+2)^{1/2} x^{1/2}}\right) \\ &\quad + (m+2)\left|\left(1 - \frac{a^{1/2}}{(m+2)^{1/2} x^{1/2}}\right)\right. \\ &\quad \left. - \left(\frac{a^{1/2}}{(m+2)^{1/2} x^{1/2}} - \frac{a^{1/2}}{(m+3)^{1/2} x^{1/2}}\right)\right| \\ &\quad + \frac{a^{1/2}}{x^{1/2}} \sum_{n=m+1}^{\infty} (n+1)\left\{\frac{1}{n^{1/2}} - \frac{1}{2(n+1)^{1/2}} + \frac{1}{(n+2)^{1/2}}\right\}.\end{aligned} \tag{140}$$

*Cf. Proceedings of the London Mathematical Society, ser. 2, vol. 4 (1906), p. 263, Theorem 2a 1. Hardy does not prove the theorem for the case where the original series has variable terms, but the necessary modifications are obvious.

†Cf. Mathematische Annalen, vol. 65 (1908), p. 361.

But we have, since $mx < a \leqq (m+1)x$

$$1 - \frac{a^{1/2}}{(m+2)^{1/2}x^{1/2}} < \frac{a^{1/2}}{m^{1/2}x^{1/2}} - \frac{a^{1/2}}{(m+2)^{1/2}x^{1/2}}$$

$$= \frac{a^{1/2}\{(m+2)^{1/2} - m^{1/2}\}}{m^{1/2}(m+2)^{1/2}x^{1/2}} < \frac{K_1}{(m+1)}, \tag{141}$$

$$\frac{1}{(m+2)^{1/2}x^{1/2}} - \frac{1}{(m+3)^{1/2}x^{1/2}} = \frac{(m+3)^{1/2} - (m+2)^{1/2}}{(m+2)^{1/2}(m+3)^{1/2}x^{1/2}} < \frac{K_2}{m+1}, \tag{142}$$

$$\frac{1}{x^{1/2}}\sum_{n=m+2}^{\infty}(n+1)\left\{\frac{1}{n^{1/2}} - \frac{1}{2(n+1)^{1/2}} + \frac{1}{(n+2)^{1/2}}\right\}$$

$$< \frac{K_3}{x^{1/2}}\sum_{n=m+2}^{\infty}\frac{1}{(n+1)^{3/2}} < \frac{K_3}{x^{1/2}}\int_{m+1}^{\infty}\frac{dx}{(x+1)^{3/2}} = \frac{K_3}{x^{1/2}(m+2)^{1/2}} < \frac{K_3}{a}, \tag{143}$$

where K_1, K_2, and K_3 are positive constants. Combining these three inequalities with (140), the inequality (139) follows at once and our lemma is proved.

17. Before proceeding to the proof of the next lemma we find it convenient to make certain reductions and introduce certain new notations. From the asymptotic expansion* for $J_\nu(\lambda x)$, equations (9), (11), and

$$\sin(\lambda_n x - \alpha) = \sin[(n\pi + q)x - \alpha] + \frac{x\psi(x,n)}{n}, \tag{144}$$

which is readily derived in the same manner as (11), we obtain

$$J_\nu(\lambda_n x) = \phi_n(x) + \frac{1}{n^{3/2}x^{1/2}}\{k_2\cos[(n\pi+q)x-\alpha] + k_4 x[\sin(n\pi+q)x-\alpha]\} + \frac{k_5}{n^{5/2}x^{3/2}}\sin[(n\pi+q)x-\alpha]$$

$$+ \frac{\psi(x,n)}{n^{5/2}x^{1/2}} + \frac{\psi(x,n)}{n^{1/2}x^{3/2}} + \frac{\psi(\lambda_n x)}{\lambda^{5/2}x^{5/2}}, \tag{145}$$

where the k's represent constants and

$$\phi_n(x) = \frac{k_1}{n^{1/2}x^{1/2}}\cos[(n\pi+q)x-\alpha] + \frac{k_3}{n^{3/2}x^{3/2}}\sin[(n\pi+q)x-\alpha].\dagger \tag{146}$$

We also set

* Cf. formula 48 of Transactions I.

† Here also the k's represent constants.

$$(147)\quad \bar{\phi}_n(x) = \frac{k_1}{a^{1/2}}\cos[(n\pi+q)x-\alpha] + \frac{k_3}{a^{3/2}}\sin[(n\pi+q)x-\alpha].$$

We are now ready for the proof of the following lemma:

LEMMA 18. *The series*

$$(148)\quad \begin{aligned}&\sum_{j=1}^{j=m+1}\bar{\phi}_j(x)\int_0^1\sqrt{y}\,\phi(y)\begin{matrix}\cos\\ \sin\end{matrix}(qy-\alpha)j\begin{matrix}\cos\\ \sin\end{matrix}j\pi y\,dy\\ &\qquad+\sum_{j=m+2}^{\infty}\phi_j(x)\int_0^1\sqrt{y}\,\phi(y)\begin{matrix}\cos\\ \sin\end{matrix}(qy-\alpha)j\begin{matrix}\cos\\ \sin\end{matrix}j\pi y\,dy\\ &\sum_{j=1}^{j=m+1}\bar{\phi}_j(x)\int_0^1\frac{\phi(y)}{\sqrt{y}}(y^2+b)\begin{matrix}\sin\\ \cos\end{matrix}(qy-\alpha)\begin{matrix}\cos\\ \sin\end{matrix}j\pi y\,dy\\ &\qquad+\sum_{j=m+2}^{\infty}\phi_j(x)\int_0^1\frac{\phi(y)}{\sqrt{y}}(y^2+b)\begin{matrix}\sin\\ \cos\end{matrix}(qy-\alpha)\begin{matrix}\cos\\ \sin\end{matrix}j\pi y\,dy,\end{aligned}$$

where m has the same significance as in Lemma 17, *will be summable* $(C\,1)$ *in the interval* $(0<x\leqq c<1)$, *provided* $\phi(y)$ *satisfies the conditions of Lemma* 15.

We will consider only the first series in (148), since the proof for the other series is analogous. It is readily seen that the former series may be obtained by introducing the functions $f_n(x)$ defined by (137) as factors of the successive terms of the series

$$(149)\quad \sum_{j=1}^{\infty}\bar{\phi}_j(x)\int_0^1\sqrt{y}\,\phi(y)\cos(qy-\alpha)j\cos j\pi y\,dy.$$

But, by Lemma 16, this latter series is uniformly summable $(C\,1)$ in the interval $(0<x\leqq c<1)$, and hence by Lemma 17 the first series in (148) is uniformly summable $(C\,1)$ in this same interval.

LEMMA 19. *The series whose general terms are*

$$(150)\quad \begin{aligned}&nJ_\nu(\lambda_n x)\int_0^1\sqrt{y}\,\phi(y)\begin{matrix}\cos(qy-\alpha)\cos n\pi y\\ \sin(qy-\alpha)\sin n\pi y\end{matrix}dy,\\ &nJ_\nu(\lambda_n x)\int_0^1\frac{\phi(y)}{\sqrt{y}}(y^2+b)\begin{matrix}\sin(qy-\alpha)\cos n\pi y\\ \cos(qy-\alpha)\sin n\pi y\end{matrix}dy,\end{aligned}$$

will be uniformly summable $(C\,\frac{1}{2})$ *in the interval* $(0<c_1\leqq x\leqq c_2<1)$, *provided* $\phi(y)$ *satisfies the conditions of Lemma* 15 *in the interval* $(0<y\leqq c_2)$.

We will consider only the first expression in (150), since the treatment of the other cases is analogous. If we substitute for $J_\nu(\lambda_n x)$ the value given by (145), the resulting expression may be written in the form

$$
\begin{aligned}
&\frac{k_1 n + k_2}{n^{1/2} x^{1/2}} \cos (qx - \alpha) \cos n\pi x \int_0^1 \sqrt{y}\, \phi(y) \cos (qy - \alpha) \cos n\pi y \, dy \\
&- \frac{k_1 n + k_2}{n^{1/2} x^{1/2}} \sin (qx - \alpha) \sin n\pi x \int_0^1 y \phi(y) \cos (qy - \alpha) \cos n\pi y \, dy \\
&+ \frac{k_3 + k_4 x^2}{n^{1/2} x^{3/2}} \{\sin (qx - \alpha) \cos n\pi x + \cos (qx - \alpha) \sin n\pi x\} \\
&\qquad \times \int_0^1 \sqrt{y}\, \phi(y) \cos (qy - \alpha) \cos n\pi y \, dy + \frac{\psi(x, n)}{n^{3/2} x^{5/2}}.
\end{aligned} \tag{151}
$$

The last term of (151) is obviously the general term of a series that is uniformly convergent in the interval in question and therefore uniformly summable $(C\,\frac{1}{2})$ there. It remains to consider the other terms.

We know from Lemmas 15, 6, and 7 that the series whose general terms are

$$
\begin{aligned}
&n \cos n\pi x \int_0^1 \left(\sqrt{y}\, \phi(y) \cos (qy - \alpha) - y \frac{\phi(x) \cos (qx - \alpha)}{\sqrt{x}} \right) \cos n\pi y \, dy, \\
&n \sin n\pi x \int_0^1 \left(\sqrt{y}\, \phi(y) \cos (qy - \alpha) - y \frac{\phi(x) \cos (qx - \alpha)}{\sqrt{x}} \right) \cos n\pi y \, dy,
\end{aligned} \tag{152}
$$

are uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c_2)$. We know from Lemmas 15 and 12 that the series whose general terms are

$$
n \cos n\pi x \int_0^1 y \cos n\pi y \, dy, \qquad n \sin n\pi x \int_0^1 y \cos n\pi y \, dy, \tag{153}
$$

are uniformly summable $(C\,1)$ in that same interval. Combining these facts and making use of the theorem of Riesz referred to in § 6, we readily infer that the first, second, and third terms of (151) are the general terms of series that are summable $(C\,\frac{1}{2})$ in the interval $(0 < c_1 \leqq x \leqq c_2)$. Since the fourth term has this same property, so also does the whole expression (151), or the first expression in (150). As the treatment of the other expressions is analogous, our lemma may be regarded as established.

Lemmas on the behavior of certain summations of terms. §§ 18–19

18. Lemma 20. *If a is the first positive root of the equation*

$$
\frac{k_1}{v^{1/2}} \cos (\pi v - \alpha) + \frac{k_3}{v^{3/2}} \sin (\pi v - \alpha) = 0, \tag{154}
$$

where k_1 and k_3 are the k's of equation (146), *then corresponding to an arbitrary, positive ϵ, an integer μ and a positive δ exist such that the expressions*

$$(155)\qquad \sum_{j=m+2}^{j=n} \frac{n-j+1}{n}\phi_j(x)\int_0^1 \sqrt{y}\,\phi(y)\begin{matrix}\cos\\ \sin\end{matrix}(qy-\alpha)j\begin{matrix}\cos\\ \sin\end{matrix}j\pi y\,dy,$$
$$\sum_{j=m+2}^{j=n} \frac{n-j+1}{n}\phi_j(x)\int_0^1 \frac{\phi(y)}{\sqrt{y}}(y^2+b)\begin{matrix}\sin\\ \cos\end{matrix}(qy-\alpha)\begin{matrix}\cos\\ \sin\end{matrix}j\pi y\,dy,$$

where $\phi_j(x)$ is defined by (146) *and m is the greatest integer such that $(0 < mx < a)$,* are less in absolute value than ϵ throughout the interval $(0 < x \leqq \delta)$ when $n \geqq \mu$, provided $\phi(y)$ satisfies the conditions of Lemma* 15.

We find it convenient to begin by deriving certain properties of the functions $\bar{\phi}_n(x)$ defined by (147), the a of that equation being taken as the a of the present lemma. Since a is a root of (154) and since $mx < a \leqq (m+1)x$, we have

$$(156)\qquad \begin{aligned}|\bar{\phi}_{m+1}(x)| \leqq \Big|\frac{2k_1}{a^{1/2}}\sin\Big[\frac{\pi\{(m+1)x+a\}+qx}{2}-\alpha\Big]\\ \times\sin\frac{\pi\{(m+1)x-a\}+qx}{2}\Big|\\ +\Big|\frac{2k_3}{a^{3/2}}\cos\Big[\frac{\pi\{(m+1)x+a\}+qx}{2}-\alpha\Big]\\ \times\sin\frac{\pi\{(m+1)x-a\}+qx}{2}\Big| < \frac{B_1}{m},\end{aligned}$$

where B_1 is a positive constant, since the second factor in each term of the second member of (156) is easily seen to be of order $1/m$. Similarly we may establish

$$(157)\qquad |\bar{\phi}_{m+2}(x)| < \frac{B_2}{m},$$

where B_2 is a positive constant.

If we set

$$(158)\qquad h(v) = \frac{k_1}{a^{1/2}}\cos(\pi v-\alpha)+\frac{k_3}{a^{3/2}}\sin(\pi v-\alpha),$$

we have from the definition of $\bar{\phi}_j(x)$

$$(159)\qquad \begin{aligned}\bar{\phi}_j(x) = h(jx) - \frac{2k_1}{a^{1/2}}\sin\left[(j\pi+\tfrac{1}{2}q)x-\alpha\right]\sin\frac{qx}{2}\\ +\frac{2k_3}{a^{3/2}}\cos\left[(j\pi+\tfrac{1}{2}q)x-\alpha\right]\sin\frac{qx}{2},\end{aligned}$$

* It should be understood that when $(m+2) > n$, there are no terms in the expressions (155).

whence

$$\Delta\bar{\phi}_j(x) = \Delta h(jx) + x^2\psi_1(x,j) \qquad \left(\begin{matrix}|\psi_1(x,j)| < B_3\\ j = 1,2,3,\cdots\end{matrix}\right), \tag{160}$$

$$\Delta^2\bar{\phi}_j(x) = \Delta^2 h(jx) + x^3\psi_2(x,j) \qquad \left(\begin{matrix}|\psi_2(x,j)| < B_4\\ j = 1,2,3,\cdots\end{matrix}\right), \tag{161}$$

B_3 and B_4 being positive constants. Furthermore we have

$$|h(jx) - h((j+1)x)| < B_5 x \qquad (j = 1,2,3,\cdots), \tag{162}$$

$$|h(jx)| < B_6 \qquad (j = 1,2,3,\cdots), \tag{163}$$

where B_5 and B_6 are positive constants.

We will give the proof only for the first expression in (155), since the discussion for the other cases is analogous. Let us represent by $S(x)$ the value to which the first series in (148) is summable, and let us consider the series obtained from that series by substituting for the first term that same term minus $S(x)$. This series will be uniformly summable $(C\,1)$ to the value zero in the interval $(0 < x \leqq c)$. Hence if we use $\epsilon_j(x)$ to represent $S_j(x)/j$, where $S_j(x)$ stands for the sum of the first j sums of j terms of this series, it is apparent that $\epsilon_j(x)$ approaches zero uniformly in this interval and therefore, corresponding to an arbitrary positive ϵ, we may choose $p > 2$ and such that

$$|\epsilon_j(x)| < \eta = \frac{\epsilon}{2\{1 + 2B_1 + B_2 + B_4 a^3 + 2(r+1)(B_5 a + B_6) + 2[B_3 a^2 + 2(r_1+1)B_6]\}} \qquad \left(\begin{matrix}j \geqq p\\ 0 < x \leqq c\end{matrix}\right), \tag{164}$$

where the B's are the B's of (156), (157), (160), (161), (162), and (163), and r and r_1 represent respectively the number of times that $h''(v)$ and $h'(v)$ change sign in the interval $(0 < v < 2a)$. Then, p being fixed, we choose $\delta < a/p$ and $< c$, and so small that

$$\left|\sum_{j=1}^{j=p}\left\{j\,\epsilon_j(x)\Delta^2\bar{\phi}_j(x) + \frac{j}{n}\epsilon_j(x)\Delta\bar{\phi}_j(x)\right\}\right| < \eta \qquad \left(\begin{matrix}0 < x \leqq \delta\\ n = 1,2,3,\cdots\end{matrix}\right), \tag{165}$$

which we may do in view of the fact that $\Delta^2\bar{\phi}_j(x)$ and $\Delta\bar{\phi}_j(x)$ approach zero with x uniformly for all values of j, and $\epsilon_j(x)$ remains finite for all values of j and all values of x in the interval $(0 < x \leqq c)$.

Since the series we are considering is uniformly summable $(C\,1)$ to zero in the interval $(0 < x \leqq c)$, the expression

$$\begin{aligned}&\left\{\bar{\phi}_1(x)\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos\pi y\,dy - S(x)\right\}\\ &+ \sum_{j=2}^{j=m+1}\frac{n-j+1}{n}\bar{\phi}_j(x)\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\,j\cos j\pi y\,dy\\ &+ \sum_{j=m+2}^{j=n}\frac{n-j+1}{n}\phi_j(x)\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\,j\cos j\pi y\,dy\end{aligned} \tag{166}$$

can be made less in absolute value than $\frac{1}{2}\epsilon$ for all values of n greater than a certain fixed integer and for all values of x in $(0 < x \leqq \delta)$. If we can show that the same is true for the sum of the first two terms of (166), it will follow that the third term can be made less in absolute value than ϵ for all values of x in $(0 < x \leqq \delta)$ and all values of n greater than a fixed integer, and our lemma will be proved.

The qth term of the expression formed from the first two terms of (166) may be replaced by $(q\epsilon_q(x) - 2(q-1)\epsilon_{q-1}(x) + (q-2)\epsilon_{q-2}(x))$, where we write $\epsilon_{-1}(x) = \epsilon_{-2}(x) = 0$. If we make this substitution, drop the parentheses, and rearrange the terms so that terms involving the same subscript for the ϵ's are brought together, we obtain

$$(167)\quad \sum_{j=1}^{j=m} j\,\epsilon_j(x)\Delta^2\left(\frac{n-j+1}{n}\bar{\phi}_j(x)\right) + (m+1)\,\epsilon_{m+1}(x)\frac{n-m}{n}\bar{\phi}_{m+1}(x) - m\,\epsilon_m(x)\frac{n-m-1}{n}\bar{\phi}_{m+2}(x).$$

Moreover, since

$$(168)\quad \Delta^2\frac{n-j+1}{n}\bar{\phi}_j(x) = \frac{2}{n}\Delta\bar{\phi}_{j+1}(x) + \frac{n-j+1}{n}\Delta^2\bar{\phi}_j(x),$$

the first term of (167) takes the form

$$(169)\quad \sum_{j=1}^{j=m}\left\{ j\,\epsilon_j(x)\frac{n-j+1}{n}\Delta^2\bar{\phi}_j(x) + \frac{2j}{n}\epsilon_j(x)\Delta\bar{\phi}_{j+1}(x)\right\}.$$

We have from (161)

$$(170)\quad \sum_{j=p+1}^{j=m} j\,\epsilon_j(x)\Delta^2\bar{\phi}_j(x) = \sum_{j=p+1}^{j=m} j\,\epsilon_j(x)\Delta^2 h(jx) + x^3\sum_{j=p+1}^{j=m} j\,\epsilon_j(x)\psi_2(x,j).$$

But

$$(171)\quad \left| x^3\sum_{j=p+1}^{m=1} j\,\epsilon_j(x)\psi_2(x,j)\right| < B_4\,\eta x^3\sum_{j=p+1}^{j=m} j < B_4\,\eta m^2 x^3 < B_4\,a^3\,\eta,$$

and if $p_1, p_2, \cdots, p_s$ represent the values of j at which $\Delta^2 h(jx)$ changes sign and we set $p+1 = p_0$ and $m = p_{s+1}$, we have from (162) and (163)

$$(172)\quad \begin{aligned}\left|\sum_{j=p+1}^{j=m} j\,\epsilon_j(x)\Delta^2 h(jx)\right| &\leqq \eta\sum_{i=0}^{i=s}\left|\sum_{j=p_i}^{j=p_{i+1}} j\,\Delta^2 h(jx)\right| \\ &= \eta\sum_{i=0}^{i=s}|\,p_i\{h(p_i x) - h((p_i+1)x)\} + h((p_i+1)x) \\ &\quad - p_{i+1}\{h((p_{i+1}+1)x) - h((p_{i+1}+2)x)\} - h((p_{i+1}+1)x)| \\ &< \eta(s+1)2(B_5\,a + B_6) \leqq 2(r+1)(B_5\,a + B_6)\,\eta,\end{aligned}$$

since obviously $s \leqq r$. Combining (170), (171), and (172), we have

$$\left| \sum_{j=p+1}^{j=m} j\,\epsilon_j(x)\,\bar{\Delta}^2\phi_j(x) \right| < \{B_4 a^3 + 2(r+1)(B_5 a + B_6)\}\eta \qquad (0 < x \leqq c). \tag{173}$$

By means of a similar discussion, making use of (160) and (163), we obtain for the second term in (169) the inequality

$$\frac{2}{n}\left| \sum_{j=p+1}^{j=m} j\,\epsilon_j(x)\,\bar{\Delta}\phi_{j+1}(x) \right| < 2\{B_3 a^2 + 2(r_1+1)B_5\}\eta \quad (0 < x \leqq c). \tag{174}$$

We have, moreover, from (156) and (157)

$$|(m+1)\,\epsilon_{m+1}(x)\,\bar{\phi}_{m+1}(x)| < 2B_1\eta, \qquad |m\epsilon_m(x)\,\bar{\phi}_{m+2}(x)| < B_2\eta. \tag{175}$$

Combining (175) with (174), (173), (169), and (165), we find that (167) is less in absolute value than $\frac{1}{2}\epsilon$ for values of x in the interval $(0 < x \leqq \delta)$. Thus, as pointed out before, our lemma is proved.

19. LEMMA 21. *Given an arbitrary positive ϵ, we can choose a μ and a δ such that the expressions*

$$\begin{aligned} &\sum_{j=m+2}^{j=n} \frac{n-j+1}{n} F_j(x) \int_0^1 \sqrt{y}\,\phi(y) \begin{matrix}\cos\\ \sin\end{matrix}(qy-\alpha)\, j \begin{matrix}\cos\\ \sin\end{matrix} j\pi y\, dy, \\ &\sum_{j=m+2}^{j=n} \frac{n-j+1}{n} F_j(x) \int_0^1 \frac{\phi(y)}{\sqrt{y}}(y^2+b) \begin{matrix}\sin\\ \cos\end{matrix}(qy-\alpha) \begin{matrix}\cos\\ \sin\end{matrix} j\pi y\, dy, \end{aligned} \tag{176}$$

where

$$F_j(x) = \frac{1}{n^{3/2}x^{1/2}}\{k_2 \cos[(n\pi+q)x-\alpha] + k_4 x \sin[(n\pi+q)x-\alpha]\}, \tag{177}$$

and m is defined as in Lemma 20, are less in absolute value than ϵ in the interval $(0 < x \leqq \delta)$ when $n \geqq \mu$, provided $\phi(y)$ satisfies the conditions of Lemma 15.*

We will carry through the proof only for the first expression in (176), since the proof for the other cases is analogous.

We begin by defining

$$\bar{F}_j(x) = \frac{1}{na^{1/2}}\{k_2 \cos[(n\pi+q)x-\alpha] + k_4 x \sin[(n\pi+q)x-\alpha]\}. \tag{178}$$

The series

$$\sum_{j=1}^{\infty} \bar{F}_j(x) \int_0^1 \sqrt{y}\,\phi(y) \cos(qy-\alpha)\, j \cos j\pi y\, dy \tag{179}$$

is by Lemma 16 and the theorem of Riesz referred to in § 6, uniformly convergent in the interval $(0 < x \leqq c)$. Hence by Lemma 17 the series

* See also footnote to the statement of this lemma.

$$(180)\qquad \sum_{j=1}^{j=m+1} \overline{F}_j(x) \int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha) j\cos j\pi y\,dy + \sum_{j=m+2}^{\infty} F_j(x) \int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha) j\cos j\pi y\,dy$$

is uniformly convergent in the interval $(0 < x \leqq c)$. We choose μ so large that in the series (180)

$$(181)\qquad \left| \sum_{j=p}^{j=n} u_j(x) \right| < \epsilon, \qquad (n > p \geqq \mu),$$

$u_j(x)$ representing the jth term of (180). Then we choose $\delta \leqq a/\mu$, so that for any point in $(0 < x \leqq \delta)$, $(m+2) \geqq a/x \geqq a/\delta \geqq \mu$. Under these conditions we have

$$(182)\qquad \left| \sum_{j=m+2}^{j=n} u_j(x) \right| < \epsilon \qquad \begin{pmatrix} 0 < x \leqq \delta \\ n \geqq \mu \end{pmatrix}.$$

From (182) and Abel's lemma, it follows that the first expression in (176) satisfies the desired inequality.

Uniform summability of the development in the neighborhood of the origin. §§ 20–21

20. Before taking up the proof of the next lemma, we find it convenient to make some further reductions. From (145), (177), the identity

$$(183)\qquad \Delta^2 g_n f_n = g_n \Delta^2 f_n + 2\Delta g_n \Delta f_{n+1} + f_{n+2} \Delta^2 g_n,$$

and the use of the Law of the Mean for derivatives, we have

$$(184)\qquad \Delta^2 J_\nu(\lambda_n x) = \Delta^2 \phi_n(x) + \Delta^2 F_n(x) + R_n(x),$$

where

$$(185)\qquad R_n(x) = \frac{\psi(x,n)}{n^{5/2} x^{1/2}} + \frac{\psi(x,n)}{n^{7/2} x^{3/2}} + \frac{\psi(x,n)}{n^{9/2} x^{5/2}}.$$

Similarly, using the identity

$$(186)\qquad \Delta g_n f_n = g_n \Delta f_n + f_{n+1} \Delta g_n,$$

we obtain

$$(187)\qquad \Delta J_\nu(\lambda_n x) = \Delta\phi_n(x) + \Delta F_n(x) + S_n(x),$$

where

$$(188)\qquad S_n(x) = \frac{\psi(x,n)}{n^{5/2} x^{3/2}} + \frac{\psi(x,n)}{n^{7/2} x^{5/2}}.$$

We will also find it convenient to derive a few preliminary inequalities that will be of use in the proof of the next lemma. From the definitions of $\phi_n(x)$ and $\overline{\phi}_n(x)$, given in (146) and (147), and the inequalities (156) and (157), we obtain for values of m and x such that $mx < a \leqq (m+1)x$, where a is defined as in Lemma 20, the following relations

$$(189) \qquad |m\phi_{m+2}(x)| < D_1, \qquad |(m+1)\phi_{m+1}(x)| < D_2,$$

where D_1 and D_2 are positive constants. Furthermore, from the definition of $F_j(x)$, given by (177), we readily obtain for values of m and x such that $mx < a \leqq (m+1)x$

$$(190) \qquad |mF_{m+2}(x)| < D_1', \qquad |(m+1)F_{m+1}(x)| < D_2',$$

where D_1' and D_2' are positive constants. Finally we have

$$(191) \qquad |\phi_n(x)| < K_1, \qquad |F_n(x)| < K_2 \qquad (nx \geqq a);$$

where K_1 and K_2 are positive constants.

LEMMA 22. *The series whose general terms are*

$$(192) \qquad \begin{aligned} &n J_\nu(\lambda_n x) \int_0^1 \sqrt{y}\,\phi(y) \begin{matrix}\cos\\ \sin\end{matrix}(qy-\alpha) \begin{matrix}\cos\\ \sin\end{matrix} n\pi y\, dy, \\ &J_\nu(\lambda_n x) \int_0^1 \frac{\phi(y)}{\sqrt{y}} (y^2+b) \begin{matrix}\sin\\ \cos\end{matrix}(qy-\alpha) \begin{matrix}\cos\\ \sin\end{matrix} n\pi y\, dy, \end{aligned}$$

will be uniformly summable $(C\,1)$ *in the interval* $(0 < x \leqq c)$, *provided* $\phi(y)$ *satisfies the conditions of Lemma* 15.

We will consider only the series whose general term is the first term in (189), since the proof for the other cases is analogous. The series whose general term is n times the integral in the first term of (189) is summable $(C\,1)$ by Lemma 2. Let S be the value to which it is summable and let us replace the first term of the series we are considering by that same term minus $SJ_\nu(\lambda_1 x)$. If we can establish the uniform summability of the resulting series in the interval $(0 < x \leqq c)$, it is thereby proved that the original series has the same property.

Let us represent by ϵ_j the expression $(S_j/j - S)$, where S_j indicates the sum of the first j sums of j terms for the series summable to S. Then, given an arbitrary positive ϵ, we choose p so large that

$$(193) \qquad |\epsilon_j| < \eta = \frac{\epsilon}{4\{1 + (2r + 2r_1 + 5)C + 2(r+1)C_1 da + \frac{4}{5}a^{-5/2}(5a^2 + 10a + 12)M + D_1 + D_1' + D_2 + D_2' + K_1 + K_2\}} \qquad (j \geqq p),$$

where r and r_1 represent the number of times that $J_\nu''(x)$ and $J_\nu'(x)$ respectively change signs in the interval $(0, 2a)$, a having the same significance as in Lemma 20, the D's and K's are as defined by (189), (190), and (191), d is the upper limit of $(\lambda_{n+1} - \lambda_n)$, M is the upper limit of the functions $\psi(x, n)$ in (185) and (188), and C and C_1 represent the maximum absolute value of $J_\nu(x)$ and $J_\nu'(x)$. Now, p being fixed, we choose $\mu_1 > p$ and a positive $\delta < c$, such that the first expressions in (155) and (176) are each less in absolute value than $\frac{1}{8}\epsilon$ throughout the interval $(0 < x \leqq \delta)$ for $n \geqq \mu_1$, and so

that

$$(194)\qquad \left|\frac{1}{n}\sum_{j=1}^{j=p-1}\epsilon_j\{(n-j+1)\,\Delta^2 J_\nu(\lambda_j x)+2\Delta J_\nu(\lambda_j x)\}\right|<\eta \quad (0<x\leqq\delta).$$

By Lemma 19 the series we are considering is uniformly summable ($C\,\frac{1}{2}$), and therefore ($C\,1$), in the interval ($\delta\leqq x\leqq c$). Hence we may choose μ_2 so large that

$$(195)\qquad |\sigma_{n_1}(x)-\sigma_{n_2}(x)|<\epsilon \qquad \begin{pmatrix} n_2>n_1\geqq\mu_2 \\ \delta\leqq x\leqq c\end{pmatrix},$$

where $\sigma_j(x)$ represents the arithmetic mean of the first j sums of the first j terms of the series in question. It remains to be shown that the same inequality holds for the interval $(0<x<\delta)$.

We have

$$\sigma_n(x)=(u_1-S)J_\nu(\lambda_1 x)+\sum_{j=2}^{j=n}\frac{n-j+1}{n}u_j J_\nu(\lambda_j x),$$

where

$$u_j=\int_0^1\sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos j\pi y\,dy.$$

Furthermore, if in the above expression for $\sigma_n(x)$ we replace u_j by its value $(\epsilon_j-2\epsilon_{j-1}+\epsilon_{j-2})$,* drop parentheses, and rearrange terms according to subscripts of ϵ, we obtain

$$(196)\qquad \sigma_n(x)=\frac{1}{n}\sum_{j=1}^{j=n}j\,\epsilon_j\{(n-j+1)\,\Delta^2 J_\nu(\lambda_j x)+2\Delta J_\nu(\lambda_{j+1}x)\}+\epsilon_n J_\nu(\lambda_{n+2}x).$$

Three cases will arise here, according as m, the greatest integer $<a/x$, where a is the a of Lemma 20, is $<p$, lies between p and n, or is $\geqq n$. We will discuss only the second case, since this discussion virtually includes the discussion of the other two cases.

If we represent by $p_1, p_2, \cdots, p_s$ the values of j between p and m for which $\Delta^2 J_\nu(\lambda_j x)$ changes sign, and we set $p=p_0$, $m=p_{s+1}$, we have

$$\begin{aligned}(197)\qquad &\left|\sum_{j=p}^{j=m}j\,\epsilon_j\frac{n-j+1}{n}\Delta^2 J_\nu(\lambda_j x)\right|\leqq\eta\sum_{i=0}^{i=s}\left|\sum_{j=p_i}^{j=p_{i+1}}j\frac{n-j+1}{n}\Delta^2 J_\nu(\lambda_j x)\right|\\ &<\eta\sum_{i=0}^{i=s}\left|\sum_{j=p_i}^{j=p_{i+1}}j\,\Delta^2 J_\nu(\lambda_j x)\right|\\ &=\eta\sum_{i=0}^{i=s}|\,p_i\{J_\nu(\lambda_{p_i}x)-J_\nu(\lambda_{p_i+1}x)\}+J_\nu(\lambda_{p_i+1}x)\\ &\quad-p_{i+1}\{J_\nu(\lambda_{p_{i+1}+1}x)-J_\nu(\lambda_{p_{i+1}+2}x)\}-J_\nu(\lambda_{p_{i+1}+1}x)|\\ &<2(s+1)(C+C_1\,da)\,\eta\leqq 2(r+1)(C+C_1\,da)\,\eta\\ &\qquad(0<x\leqq\delta).\end{aligned}$$

* In the case of the first term we replace (u_1-S) by $\epsilon_1-2\epsilon_{-1}+\epsilon_{-2}$. It is understood, of course, that $\epsilon_{-1}=\epsilon_{-2}=0$.

Also, if we represent by $p'_1, p'_2, \cdots, p'_{s_1}$, the values of j between p and m at which $\Delta J_\nu(\lambda_{j+1}x)$ changes sign, and we set $p'_0 = p$, $p'_{s_1+1} = m$, we have

$$\left|\sum_{j=p}^{j=m}\frac{j}{n}\epsilon_j\Delta J_\nu(\lambda_{j+1}x)\right| < \eta\sum_{i=0}^{i=s_1}\left|\sum_{j=p'_i}^{j=p'_{i+1}}\Delta J_\nu(\lambda_{j+1}x)\right|$$

$$= \eta\sum_{i=0}^{i=s_1}\left|J_\nu(\lambda_{p'_i+1}x) - J_\nu(\lambda_{p'_{i+1}+1}x)\right| \tag{198}$$

$$< 2(s_1+1)C\eta \leqq 2(r_1+1)C\eta \qquad (0 < x \leqq \delta).$$

Moreover, we have from (184) and (187)

$$\frac{1}{n}\sum_{j=m+1}^{j=n} j\,\epsilon_j\{(n-j+1)\Delta^2 J_\nu(\lambda_j x) + 2\Delta J_\nu(\lambda_{j+1}x)\}$$

$$= \frac{1}{n}\sum_{j=m+1}^{j=n} j\,\epsilon_j\{(n-j+1)[\Delta^2\phi_j(x) + \Delta^2 F_j(x)] \tag{199}$$

$$+ 2[\Delta\phi_{j+1}(x) + \Delta F_{j+1}(x)]\}$$

$$+ \sum_{j=m+1}^{j=n}\frac{n-j+1}{n}\epsilon_j\, j\, R_j(x) + 2\sum_{j=m+2}^{j=n}\epsilon_j\frac{j}{n}S_j(x).$$

But, from (185)

$$\left|\sum_{j=m+1}^{j=n}\frac{n-j+1}{n}\epsilon_j\, j\, R_j(x)\right| < \eta M\sum_{j=m+2}^{j=n}\left[\frac{1}{j^{3/2}x^{1/2}} + \frac{1}{j^{5/2}x^{3/2}} + \frac{1}{j^{7/2}x^{5/2}}\right]$$

$$< \eta M\left[\frac{4}{(m+1)^{1/2}x^{1/2}} + \frac{8}{3(m+1)^{3/2}x^{3/2}} + \frac{16}{5(m+1)^{5/2}x^{5/2}}\right] \tag{200}$$

$$< \frac{4(15a^2+10a+12)M}{15a^{5/2}}\eta.$$

Also, from (188),

$$\left|2\sum_{j=m+1}^{j=n}\epsilon_j\frac{j}{n}S_j(x)\right| < 2\eta M\sum_{j=m+2}^{j=n}\left[\frac{1}{j^{5/2}x^{3/2}} + \frac{1}{j^{7/2}x^{5/2}}\right]$$

$$< 2\eta M\left[\frac{8}{3(m+1)^{3/2}x^{3/2}} + \frac{16}{5(m+1)^{5/2}x^{5/2}}\right] < \frac{16(5a+6)M}{15a^{5/2}}\eta. \tag{201}$$

It remains to consider the first summation on the right-hand side of (199). If we rearrange this summation according to subscripts of ϕ and F, it takes the form

$$\sum_{j=m+2}^{j=n}\frac{n-j+1}{n}\{\phi_j(x) + F_j(x)\}\int_0^1\sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos j\pi y\,dy$$

$$-\frac{n-m-1}{n}m\epsilon_m\{\phi_{m+2}(x) + F_{m+2}(x)\} \tag{202}$$

$$+\frac{n-m}{n}(m+1)\epsilon_{m+1}\{\phi_{m+1}(x) + F_{m+1}(x)\}$$

$$-\epsilon_n\{\phi_{n+2}(x) + F_{n+2}(x)\}.$$

We have already chosen μ_1 and δ so that for values of x in the interval $(0 < x \leqq \delta)$ and values of $n \geqq \mu_1$ the summation in (202) should be less in absolute value than $\frac{1}{4}\epsilon$. Hence, making use of the inequalities (189), (190), and (191), we see that the expression (202) is for values of x in the interval $(0 < x \leqq \delta)$ and values of $n \geqq \mu_1$, less in absolute value than

$$\tfrac{1}{4}\epsilon + (D_1 + D_1' + D_2 + D_2' + K_1 + K_2)\eta.$$

Combining this fact with (196), (194), (197), (198), (199), (200), and (201), we obtain

$$|\sigma_n(x)| < \frac{\epsilon}{4} + \{1 + (2r + 2r_1 + 5)C + 2(r+1)C_1 da + \tfrac{4}{5}a^{-5/2}(5a^2 + 10a + 12)M + D_1 + D_1' + D_2 + D_2' + K_1 + K_2\}\eta = \frac{\epsilon}{2} \qquad \begin{pmatrix} 0 < x \leqq \delta \\ n \geqq \mu_1 \end{pmatrix},$$

or

$$|\sigma_{n_1}(x) - \sigma_{n_2}(x)| < \epsilon \qquad \begin{pmatrix} 0 < x \leqq \delta \\ n_2 > n_1 \geqq \mu_1 \end{pmatrix}. \tag{203}$$

Combining (203) with (195), we have

$$|\sigma_{n_1}(x) - \sigma_{n_2}(x)| < \epsilon \qquad \begin{pmatrix} 0 < x \leqq c \\ n_2 > n_1 \geqq \mu \end{pmatrix}, \tag{204}$$

where we have chosen for μ the larger of μ_1 and μ_2. Our lemma is therefore proved.

21. We are now ready to prove the following theorem:

THEOREM II. *If $f(x)$ is such that $\phi(x) = f(x) - f(0)$ satisfies the conditions of Lemma 15, the development of $f(x)$ in Bessel's functions of order ν $(\nu \geqq 0)$ will be uniformly summable $(C\,\frac{1}{2})$ in the interval $(0 \leqq x \leqq c)$, provided $\nu = 0$ or $f(0) = 0$.*

In view of Lemma 1 the general term of the development (12) may be written in the form

$$\left(M_1\sqrt{n} + \frac{M_2}{\sqrt{n}}\right) J_\nu(\lambda_n x) \int_0^1 \sqrt{x}\,\phi(x) \cos\{(n\pi + q)x - \alpha\}dx$$

$$+ \frac{M_3}{\sqrt{n}} J_\nu(\lambda_n x) \int_0^1 \frac{\phi(x)}{\sqrt{x}} (x^2 + b) \sin\{(n\pi + q)x - \alpha\}dx \tag{205}$$

$$+ (-1)^n M_4 f(0) \frac{J_\nu(\lambda_n x)}{\sqrt{n}} + \frac{\nu K f(0)}{\lambda_n} J_\nu(\lambda_n x) + r_n J_\nu(\lambda_n x),$$

where r_n is the general term of an absolutely convergent series. We see at once from Lemmas 22, 2, and 3, and the theorem of M. Riesz referred to in the proof of Theorem I, that the first two terms of (205) are the general terms

of series that are uniformly summable ($C\frac{1}{2}$) in the interval ($0 \leqq x \leqq c$). The last term is obviously the general term of a series that has the same property and the fourth term is zero under the conditions of our theorem. The uniform convergence in ($0 \leqq x \leqq c$) of the series of which the third term is the general term has been established in the course of the proof of Lemma 2 of Transactions II. For if we take $c = 1$ in the first expression of (51) of that paper, this expression reduces to the third term of (205), except for a constant factor.

Since each term of (205) is the general term of a series that is uniformly summable ($C\frac{1}{2}$) in ($0 \leqq x \leqq c$), the whole expression has the same property, and our theorem is proved.

VALUE OF THE DEVELOPMENT. §§ 22–24

22. We will now show that under the conditions imposed on $f(x)$ in the previous sections the value to which the development is summable at the origin and uniformly summable in the neighborhood of the origin, will be $f(x)$. We begin by proving some lemmas.

LEMMA 23. *If the function $F(x)$ has a Lebesgue integral in the interval ($0 \leqq x \leqq \pi$), the series whose general terms are*

$$\frac{1}{n}\left\{\begin{matrix}\cos\\ \sin\end{matrix} nx \int_0^\pi F(x) \begin{matrix}\sin\\ \cos\end{matrix} nx\, dx\right\}$$

$$\frac{1}{n}\left\{\begin{matrix}\cos\\ \sin\end{matrix} nx \int_0^\pi F(x) \begin{matrix}\cos\\ \sin\end{matrix} nx\, dx\right\} \tag{206}$$

will be summable ($C, k > 0$) almost everywhere in the interval ($0 \leqq x \leqq \pi$) and may be multiplied by a function of bounded variation and integrated term by term in this interval.

Since, by virtue of the Riemann-Lebesgue theorem* the integrals in (206) approach zero as n becomes infinite, the expressions (206) are of the form $a_n \cos nx$, $b_n \sin nx$, where

$$\lim_{n\to\infty} na_n = 0 = \lim_{n\to\infty} nb_n.$$

Hence the various terms of (206) are the general terms of the Fourier's series of summable functions, and consequently in view of well-known theorems are summable ($C, k > 0$) almost everywhere† and may be multiplied by a function of bounded variation and integrated term by term over any interval.‡

* See footnote, § 7.

† Cf. Hardy, Proceedings of the London Mathematical Society, ser. 2, vol. 12 (1913), p. 365.

‡ Cf. Young, Proceedings of the London Mathematical Society, ser. 2, vol. 9 (1910), p. 449.

LEMMA 24. *The series whose general term is*

$$\begin{aligned}(207)\quad &\cos(qx-\alpha)\cos n\pi x\int_0^1\sqrt{x}f(x)\sin(qx-\alpha)\sin n\pi x\,dx\\ &+\sin(qx-\alpha)\sin n\pi x\int_0^1\sqrt{x}f(x)\cos(qx-\alpha)\cos n\pi x\,dx,\end{aligned}$$

where $f(x)$ is integrable (Lebesgue) in the interval $(0,\pi)$ and q and α have the same significance as in equation (15), will be uniformly convergent in the interval $(0\leqq x\leqq\pi)$.

The expression (207) may be written in the form (cf. Transactions I, page 412).

$$\begin{aligned}(208)\quad &\frac{1}{2}\int_0^1\sqrt{x'}f(x')\sin[q(x+x')-2\alpha]\sin n\pi(x+x')\,dx'\\ &+\frac{1}{2}\int_0^1\sqrt{x'}f(x')\sin q(x-x')\sin n\pi(x-x')\,dx'.\end{aligned}$$

Since, however,

$$\begin{aligned}\sum_1^{n=n}\sin mz&=\frac{\sin z-\sin(n+1)z+\sin nz}{2(1-\cos z)}\\ &=\frac{\sin z}{4\sin^2\frac{1}{2}z}-\frac{\cos(n+\frac{1}{2})z}{4\sin\frac{1}{2}z},\end{aligned}$$

the sum of the first n terms of the series whose general term is the second term of (208) may be written in the form

$$\begin{aligned}&\frac{1}{8}\int_0^1\sqrt{x'}f(x')\frac{\sin q(x-x')\sin\pi(x-x')}{\sin^2\frac{1}{2}\pi(x-x')}\,dx'\\ (210)\quad &-\frac{1}{8}\int_0^1\sqrt{x'}f(x')\frac{\sin q(x-x')}{\sin\frac{1}{2}\pi(x-x')}\cos(n+\tfrac{1}{2})x\cos(n+\tfrac{1}{2})x'\,dx'\\ &+\frac{1}{8}\int_0^1\sqrt{x'}f(x')\frac{\sin q(x-x')}{\sin\frac{1}{2}\pi(x-x')}\sin(n+\tfrac{1}{2})x\sin(n+\tfrac{1}{2})x'\,dx'.\end{aligned}$$

The first term of this expression is independent of n, and in view of Lemma 4, the second and third terms approach zero uniformly in $(0\leqq x\leqq 1)$ as n becomes infinite. Hence the series whose general term is the second term of (208) is uniformly convergent in this interval. It remains to prove the same for the series whose general term is the first term in (208).

From (209) we have for the sum of the first n terms of this series

$$\begin{aligned}&\frac{1}{8}\int_0^1\sqrt{x'}f(x')\frac{\sin[q(x+x')-2\alpha]\sin\pi(x+x')}{\sin^2\frac{1}{2}\pi(x+x')}\,dx'\\ &-\frac{1}{8}\int_0^1\sqrt{x'}f(x')\frac{\sin[q(x+x')-2\alpha]}{\sin\frac{1}{2}\pi(x+x')}\cos(n+\tfrac{1}{2})x\cos(n+\tfrac{1}{2})x'\,dx'\\ &+\frac{1}{8}\int_0^1\sqrt{x'}f(x')\frac{\sin[q(x+x')-2\alpha]}{\sin\frac{1}{2}\pi(x+x')}\sin(n+\tfrac{1}{2})x\sin(n+\tfrac{1}{2})x'\,dx'.\end{aligned}$$

This expression may be shown to approach a limit uniformly in $(0 \leqq x \leqq 1)$ as n becomes infinite, in the same way that this property was proved for (210), provided we know that the expression $\sin[q(x+x') - 2\alpha]/\sin \frac{1}{2}\pi(x+x')$ approaches a finite limit as $x + x'$ approaches 2. But this has been shown to be the case on page 415 of Transactions I. Our lemma is therefore proved.

23. LEMMA 25. *If $f(x)$ satisfies the conditions of Lemma* 1, *the series* (12) *is summable $(C, k > 0)$ almost everywhere in $(0 < x \leqq 1)$ and may be multiplied by a function of bounded variation and integrated term by term over any interval lying in this interval.*

Since $f(x)$ satisfies the conditions of Lemma 1, A_n may be replaced by the expression on the right-hand side of (15). Making this substitution for A_n and replacing $J_\nu(\lambda_n x)$ by its value as given by (145) and (146), we obtain for the general term of (12) an expression which, in view of Lemmas 23 and 24 and the theorems about the Fourier's development of a summable function referred to in the last footnotes, is the general term of a series that is summable $(C, k > 0)$ almost everywhere in the interval $(0 < x \leqq 1)$ and can be multiplied by a function of bounded variation and integrated term by term over any interval in this interval. Thus our lemma is proved.

LEMMA 26. *If $f(x)$ satisfies the conditions of Theorem II, the series*

$$\sum_{n=1}^{\infty} A_n \frac{J_\nu(\lambda_n x)}{\lambda_n} \tag{211}$$

will converge uniformly in the interval $(0 \leqq x \leqq c)$, provided $\nu = 0$ or $f(0) = 0$.

We know from Theorem II that under the conditions of the present lemma the series whose general term is $A_n J_\nu(\lambda_n x)$ will be uniformly summable $(C, \frac{1}{2})$ in the interval $(0 \leqq x \leqq c)$. Since, in view of (9), we have

$$\frac{1}{\lambda_n} = \frac{1}{n\pi} + \frac{\psi_1(n)}{n^2},$$

it follows readily from this fact, the theorem of M. Riesz used in the proof of Theorem I and the expression for A_n obtained in Lemma 1, that the series (211) is uniformly convergent in the same interval.

In order to avoid the appearance of an exceptional case when the l and h of equation (1) are such that $l\nu + h = 0$, we follow the method employed by us on previous occasions* and replace the series (12) by the series

$$\sum_{n=1}^{\infty} \bar{A}_n F_\nu(\lambda_n x), \tag{212}$$

where, as before,

* Cf., for example, Transactions I, pages 418–421.

$$F_\nu(\lambda, x) = \frac{1}{\lambda^\nu} J_\nu(\lambda x), \qquad \bar{A}_n = \lambda_n^\nu A_n, \tag{213}$$

and the λ's are the roots, positive or zero, of the equation

$$\left[l \frac{\partial}{\partial x} F_\nu(\lambda, x) + h F_\nu(\lambda, x) \right]_{x=1} = 0, \tag{214}$$

arranged in increasing order of magnitude. Since the series (212) is identical with the series (12) except when $l\nu + h = 0$, and in that case only differs from it in the possession of an extra term that is finite and continuous,* all that we have proved with regard to the summability, uniform summability and term by term integrability with regard to (12), holds good with regard to (212). Hence we infer from Lemma 25 that if $f(x)$ satisfies the conditions of Lemma 1, the series (212) will be summable $(C, k > 0)$ almost everywhere in $(0 < x \leqq 1)$ and may be multiplied by a function of bounded variation and integrated term by term over any interval in that interval.

We are now ready to prove the following lemma:

LEMMA 27. *If $f(x)$ satisfies the conditions of Theorem II and $\chi(x)$ represents the function to which the series* (212) *is summable $(C, k > 0)$ almost everywhere in the interval $(0 < x \leqq 1)$, we have*

$$\int_0^1 x\, \chi(x) F_\nu(\lambda_k, x)\, dx = \int_0^1 x f(x) F_\nu(\lambda_k, x)\, dx \quad (k = 1, 2, 3, \cdots). \tag{215}$$

Since, as we have just pointed out, the series (212) may be multiplied by a function of bounded variation and integrated term by term, we have the equalities

$$\int_\delta^1 x\, \chi(x) F_\nu(\lambda_k, x)\, dx = \sum_{n=1}^{\infty} \bar{A}_n \int_\delta^1 x F_\nu(\lambda_n, x) F_\nu(\lambda_k, x)\, dx \quad (k = 1, 2, 3, \cdots), \tag{216}$$

where δ is any positive number < 1. But we have (cf. Transactions I, equation (113)) for all values of $n \neq k$

$$\int_\delta^1 x F_\nu(\lambda_n, x) F_\nu(\lambda_k, x)\, dx = \frac{1}{\lambda_n^2 - \lambda_k} \left[x \{ \lambda_n^2 F_\nu(\lambda_k, x) F_{\nu+1}(\lambda_n, x) - \lambda_k^2 F_\nu(\lambda_n, x) F_{\nu+1}(\lambda_k, x) \} \right]_\delta^1. \tag{217}$$

Moreover, since equation (1) may be thrown into the form $(\nu l + h) J_\nu(\lambda) = l\lambda J_{\nu+1}(\lambda)$ (cf. Transactions I, page 397), it follows that $\lambda J_{\nu+1}(\lambda)/J_\nu(\lambda)$ is constant for all values of λ which are roots of (214) and hence we see from (213) that the right-hand side of (217) vanishes at the upper limit. At the

* L. c., page 421.

lower limit it takes the form

$$\frac{\delta}{\lambda_n^{\nu}\lambda_k^{\nu}(\lambda_n^2-\lambda_k^2)}[J_{\nu}(\lambda_k\delta)\lambda_n J_{\nu+1}(\lambda_n\delta)-\lambda_k J_{\nu+1}(\lambda_k\delta)J_{\nu}(\lambda_n\delta)], \tag{218}$$

and since from (213) $\bar{A}_n=\lambda_n^{\nu}A_n$, and from Lemma 1, $A_n=\sqrt{n}\,\psi_1(n)$, the general term of the series on the right-hand side of (216) may be written in the form

$$\frac{\delta}{\lambda_k^{\nu}}\left[J_{\nu}(\lambda_k\delta)A_n\frac{J_{\nu+1}(\lambda_n\delta)}{\lambda_n}\right]+\frac{\psi_1(n)}{n^{3/2}}. \tag{219}$$

We know from Lemma 26 that the expression (219) is the general term of a series that is uniformly convergent in the interval $(0\leqq x\leqq c)$. Hence the series on the right-hand side of (216) converges uniformly in this interval, and therefore we may let δ approach zero. But in view of the fact that for the different values of λ_k that enter, the functions $\sqrt{x}\,F_{\nu}(\lambda_k,x)$ are orthogonal to each other, all the terms but one on the right-hand side of (216) drop out, and we obtain equation (215).

24. We are now ready to prove the following theorem:

THEOREM III. *If $f(x)$ is such that $\phi(x)=f(x)-f(0)$ satisfies the conditions of Lemma* 15, *and if $\lambda_1,\lambda_2,\lambda_3,\cdots$ are the roots, positive or zero, of equation* (214) *arranged in increasing order of magnitude, then the series* (212) *will be uniformly summable $(C,\frac{1}{2})$ to $f(x)$ in the interval $(0\leqq x\leqq c)$, provided $\nu=0$ or $f(0)=0$.*

We know from Theorem II that under the conditions of the present theorem, the series (12), and therefore the series (212), is uniformly summable $(C,\frac{1}{2})$ in the interval in question. It remains to be shown that its value there is $f(x)$.

We have from Lemma 27

$$\int_0^1 x\{\chi(x)-f(x)\}F_{\nu}(\lambda_k x)\,dx=0 \qquad (k=1,2,3,\cdots), \tag{220}$$

where $\chi(x)$ is the function to which (212) is summable $(C,k>0)$ almost everywhere in the interval $(0<x\leqq 1)$. If we add to the definition of $\chi(x)$ by supposing it to be equal to the value to which the series (212) is summable $(C,\frac{1}{2})$ at any points in the interval $(0\leqq x\leqq c)$ at which the series is not summable $(C,k>0)$, the above equality is not altered, since we have at most modified the integrand at points forming a set of measure zero. But this further definition will serve to make $\chi(x)$ continuous in the interval $(0\leqq x\leqq c)$, since the series (212) is uniformly summable $(C,\frac{1}{2})$ in that interval. Hence $\chi(x)-f(x)$ is continuous there, and in view of a lemma

proved by the writer in a previous paper* $\chi(x) - f(x) = 0$, or $\chi(x) = f(x)$, at all points of the interval $(0 \leqq x \leqq c)$. Our theorem is therefore proved.

*Cf. Bulletin of the American Mathematical Society, 2d series, vol. 23 (1916), p. 25.

UNIVERSITY OF CINCINNATI,
CINCINNATI, OHIO.

ONE PARAMETER FAMILIES AND NETS OF RULED SURFACES AND A NEW THEORY OF CONGRUENCES*

BY

E. J. WILCZYNSKI

INTRODUCTION

The analytic basis for any projective theory of congruences, in which the lines of the congruence are defined either by a pair of points or a pair of planes, consists of the invariant theory of a completely integrable system of four homogeneous linear partial differential equations with two dependent and two independent variables, two of the equations of the system being of the first order, and two of the second order. If the developables of the congruence are known, this system of differential equations can be written in a very simple form.† But the determination of the developables of a congruence requires the integration of two partial differential equations of the first order, and it seems highly desirable to possess a theory which will be immediately applicable to any congruence, whether its developables can actually be found explicitly or not. The considerations made by G. M. Green‡ show that the existing theory can actually be modified so as to cover all such cases. However there are various ways in which this can be done and the various methods which might be used are not all equally desirable. Green himself has indicated one such method for the theory of congruences.§ But Green only indicated in a general way what was to be done without actually working out a complete theory. The great disadvantage which this particular theory would have, as compared with the one to which we are devoting this paper, is that only the final results would be of interest. The various types of invariants corresponding to the transformations of certain subgroups would

* Presented to the Society, December 30, 1919.

† E. J. Wilczynski, *Sur la théorie générale des congruences.* Mémoire conronnée par la classe des sciences. Mémoires publiées par la classe des Sciences de l'Académie Royale de Belgique. Collection en 4°. Deuxième séries. Tome III (1911). This paper will hereafter be cited as the Brussels Paper.

‡ G. M. Green, *On the theory of curved surfaces and canonical systems in projective differential geometry*, these Transactions, vol. 16 (1915).

§ G. M. Green, *Projective differential geometry of one-parameter families of space curves and conjugate nets on a curved surface*, American Journal of Mathematics, vol. 37 (1915), and vol. 38 (1916).

have no geometric significance. There exists another method, worked out by J. M. Kinney* in an as yet unpublished thesis, which labors under the same disadvantage.

The point of view of the present paper is to think of a congruence as being generated by a one-parameter family of ruled surfaces. If v is the parameter which varies from one of these surfaces to another, we consider first the semi-invariants and invariants of the individual ruled surfaces $v = \text{const.}$ These can be written down at once from the known theory of ruled surfaces. Let u be the variable which picks out a particular generator of such a ruled surface $v = \text{const.}$ If we now allow v to be variable while u remains constant we obtain a second one-parameter family of ruled surfaces whose generators belong to the same congruence of lines. The ruled surfaces $u = \text{const.}$ and $v = \text{const.}$ together constitute a *net* of ruled surfaces, and we shall show next how to find the invariants of this net. In terms of these we shall then obtain the invariants of the one-parameter family of ruled surfaces $v = \text{const.}$ They are those invariants of the net which do not change when the second family of the net is changed arbitrarily. Finally the invariants of the congruence are those invariants of the one-parameter family of ruled surfaces $v = \text{const.}$, which are not changed when we replace this family by any other one-parameter family of the congruence.

1. The differential equations of the problem

Let $y^{(k)}$ and $z^{(k)}$, $(k = 1, 2, 3, 4)$, be the homogeneous coördinates of two distinct points, P_y and P_z, of space. Let $y^{(k)}$ and $z^{(k)}$ be given as analytic functions of two independent variables

$$y^{(k)} = f^{(k)}(u, v), \qquad z^{(k)} = g^{(k)}(u, v) \qquad (k = 1, 2, 3, 4), \tag{1}$$

and let us regard as *corresponding* points P_y and P_z, those which correspond to the same pair of values u, v. Unless the ratios of the $y^{(k)}$'s and also the ratios of the $z^{(k)}$'s reduce to functions of the same function t of u and v, the system of lines l obtained by joining all pairs of corresponding points will form a congruence. If in (1) we put $v = \text{const.}$ we obtain a ruled surface of this congruence. Let us assume that the ruled surfaces $v = \text{const.}$ are not developable; then the determinant

$$D = |y_u^{(k)}, z_u^{(k)}, y^{(k)}, z^{(k)}| \tag{2}$$

will not vanish identically,† and we can find a unique system of differential equations of the form

* J. M. Kinney, *On the general theory of congruences without preliminary integration.*

† It is not necessary to assume that *none* of the ruled surfaces $v = \text{const.}$ are developable.

$$
\begin{aligned}
&y_{uu} + p_{11}\, y_u + p_{12}\, z_u + q_{11}\, y + q_{12}\, z = 0, \\
&z_{uu} + p_{21}\, y_u + p_{22}\, z_u + q_{21}\, y + q_{22}\, z = 0,
\end{aligned} \tag{3}
$$

satisfied by the four pairs of functions $(y^{(k)}, z^{(k)})$. The coefficients p_{ik} and q_{ik} can easily be expressed in terms of the functions $f^{(k)}$ and $g^{(k)}$, since we are assuming that D is not equal to zero. Of course p_{ik} and q_{ik} will be functions of v as well as of u. The projective properties of any individual ruled surface $v = \text{const.}$ can be completely expressed in terms of the invariants of (3).* But the same thing is not true of the properties of the whole one-parameter family of surfaces. In fact equations (3) are satisfied also by the pairs of functions

$$
\eta^{(k)} = \sum_{i=1}^{4} V_{ki}(v)\, y^{(i)}, \qquad \zeta^{(k)} = \sum_{i=1}^{4} V_{ki}(v)\, z^{(i)} \quad (k = 1, 2, 3, 4),
$$

where $V_{ki}(v)$ are arbitrary functions of v. For each constant value of v these equations represent a projective transformation of the corresponding ruled surface $v = \text{const.}$; but this transformation is in general different for different ones of these surfaces, and does not represent a projective transformation of the one-parameter family.

Still starting from (1) and the assumption $D \neq 0$, we see that we can find coefficients a_{ik} and b_{ik}, so that the four pairs of functions $y^{(k)}, z^{(k)}$ will satisfy the following equations

$$
\begin{aligned}
y_v &= a_{11}\, y_u + a_{12}\, z_u + b_{11}\, y + b_{12}\, z, \\
z_v &= a_{21}\, y_u + a_{22}\, z_u + b_{21}\, y + b_{22}\, z,
\end{aligned} \tag{4}
$$

as well as (3). It is easy to see that the most general pair of analytic functions which satisfies both of these systems of equations is

$$
y = \sum_{k=1}^{4} c_k\, y^{(k)}, \qquad z = \sum_{k=1}^{4} c_k\, z^{(k)}, \tag{5}
$$

where $c_1, \cdots, c_4$ are arbitrary constants. Consequently the system of four equations, composed of (3) and (4), may serve as basis for a projective theory of the one-parameter family of ruled surfaces $v = \text{const.}$

We have shown how to find such a system of partial differential equations for any one-parameter family of non-developable ruled surfaces, and it is evident that the resulting system will be completely integrable. It is also evident that any completely integrable system of this sort will define, except

* E. J. Wilczynski, *Projective differential geometry of curves and rules surfaces*, Leipzig, 1906, p. 133. This book will hereafter be quoted as Proj. Diff. Geom.

for projective transformations, a one-parameter family of non-developable ruled surfaces. The modifications which become necessary for the case of a one-parameter family of developables will become apparent later. We formulate our results as follows:

THEOREM. *Any analytic one-parameter family of non-developable ruled surfaces may be studied by means of a completely integrable system of partial differential equations of the form*

$$(S)\qquad \begin{aligned} y_{uu} + p_{11}\,y_u + p_{12}\,z_u + q_{11}\,y + q_{12}\,z &= 0,\\ z_{uu} + p_{21}\,y_u + p_{22}\,z_u + q_{21}\,y + q_{22}\,z &= 0,\\ y_v = a_{11}\,y_u + a_{12}\,z_u + b_{11}\,y + b_{12}\,z,&\\ z_v = a_{21}\,y_u + a_{22}\,z_u + b_{21}\,y + b_{22}\,z;& \end{aligned}$$

and every completely integrable system of this form defines a one-parameter family of non-developable ruled surfaces except for projective transformations. The notation is so chosen that the individual surfaces of the one-parameter family are obtained by equating v to a constant.

2. THE INTEGRABILITY CONDITIONS OF SYSTEM (S)

If the coefficients p_{ik}, q_{ik}, a_{ik}, b_{ik}, of (S) are chosen as arbitrary functions of u and v, the system will not be a completely integrable one. We find it necessary to obtain the integrability conditions for this system. From the last two equations of (S) we find, by differentiation and making use of the first two equations, the following expressions:

$$(6)\qquad \begin{aligned} y_{uv} &= c_{11}\,y_u + c_{12}\,z_u + d_{11}\,y + d_{12}\,z,\\ z_{uv} &= c_{21}\,y_u + c_{22}\,z_u + d_{21}\,y + d_{22}\,z,\\ y_{vv} &= e_{11}\,y_u + e_{12}\,z_u + f_{11}\,y + f_{12}\,z,\\ z_{vv} &= e_{21}\,y_u + e_{22}\,z_u + f_{21}\,y + f_{22}\,z, \end{aligned}$$

where

$$(7)\qquad \begin{aligned} c_{11} &= (a_{11})_u - a_{11}\,p_{11} - a_{12}\,p_{21} + b_{11}, & d_{11} &= (b_{11})_u - a_{11}\,q_{11} - a_{12}\,q_{21},\\ c_{12} &= (a_{12})_u - a_{11}\,p_{12} - a_{12}\,p_{22} + b_{12}, & d_{12} &= (b_{12})_u - a_{11}\,q_{12} - a_{12}\,q_{22},\\ c_{21} &= (a_{21})_u - a_{21}\,p_{11} - a_{22}\,p_{21} + b_{21}, & d_{21} &= (b_{21})_u - a_{21}\,q_{11} - a_{22}\,q_{21},\\ c_{22} &= (a_{22})_u - a_{21}\,p_{12} - a_{22}\,p_{22} + b_{22}, & d_{22} &= (b_{22})_u - a_{21}\,q_{12} - a_{22}\,q_{22}, \end{aligned}$$

and

$$
\begin{aligned}
e_{11} &= (a_{11})_v + a_{11}c_{11} + a_{12}c_{21} + b_{11}a_{11} + b_{12}a_{21},\\
e_{12} &= (a_{12})_v + a_{11}c_{12} + a_{12}c_{22} + b_{11}a_{12} + b_{12}a_{22},\\
e_{21} &= (a_{21})_v + a_{21}c_{11} + a_{22}c_{21} + b_{21}a_{11} + b_{22}a_{21},\\
e_{22} &= (a_{22})_v + a_{21}c_{12} + a_{22}c_{22} + b_{21}a_{12} + b_{22}a_{22},\\
f_{11} &= (b_{11})_v + a_{11}d_{11} + a_{12}d_{21} + b_{11}^2 + b_{12}b_{21},\\
f_{12} &= (b_{12})_v + a_{11}d_{12} + a_{12}d_{22} + b_{11}b_{12} + b_{12}b_{22},\\
f_{21} &= (b_{21})_v + a_{21}d_{11} + a_{22}d_{21} + b_{21}b_{11} + b_{22}b_{21},\\
f_{22} &= (b_{22})_v + a_{21}d_{12} + a_{22}d_{22} + b_{21}b_{12} + b_{12}^2.
\end{aligned}
\tag{8}
$$

These expressions for y_{uv}, z_{uv}, y_{vv}, z_{vv}, as well as the expressions for y_{uu} and z_{uu} obtained from (S) in terms of y_u, z_u, y, z, are determined uniquely. But if we proceed to calculate the third order derivatives of y and z, we see that some of these may be calculated in more than one way, and it becomes necessary to impose the condition that the two values obtained in this way shall be consistent. Of course the values of y_{uuu} and y_{vvv} can be obtained from (S) in one way only. The value of y_{uvv} may be obtained from either of the two equations

$$y_{uvv} = \frac{\partial y_{uv}}{\partial v}, \qquad y_{uvv} = \frac{\partial y_{vv}}{\partial u}.$$

But since y_{uv} and y_{vv}, as given by (6), were themselves obtained by differentiation of the same expression y_v contained in (S), we obtain no conditions when the two values of y_{uvv} are equated. The same remark applies to the two values of z_{uvv}. We do however obtain conditions upon the coefficients of (S) when we demand that

$$y_{uuv} = \frac{\partial y_{uu}}{\partial v} = \frac{\partial y_{uv}}{\partial u}, \qquad z_{uuv} = \frac{\partial z_{uu}}{\partial v} = \frac{\partial z_{uv}}{\partial u}.$$

The conditions obtained in this way are of the form

$$\alpha y_u + \beta z_u + \gamma y + \delta z = 0.$$

If they are to be satisfied identically, that is, for all solutions of system (S), the coefficients α, β, γ, δ, must all be equal to zero, since we are assuming that the determinant D, given by (2), is not equal to zero. Thus each of the two conditions splits up into four relations between the coefficients of (S). We obtain in this way the following eight equations; *the integrability conditions for the system* (S):

$$
\begin{aligned}
&(p_{11})_v + p_{11}c_{11} + p_{12}c_{21} + q_{11}a_{11} + q_{12}a_{21} \\
&\qquad + (c_{11})_u - c_{11}p_{11} - c_{12}p_{21} + d_{11} = 0, \\
&(p_{12})_v + p_{11}c_{12} + p_{12}c_{22} + q_{11}a_{12} + q_{12}a_{22} \\
&\qquad + (c_{12})_u - c_{11}p_{12} - c_{12}p_{22} + d_{12} = 0, \\
&(p_{21})_v + p_{21}c_{11} + p_{22}c_{21} + q_{21}a_{11} + q_{22}a_{21} \\
&\qquad + (c_{21})_u - c_{21}p_{11} - c_{22}p_{21} + d_{21} = 0, \\
&(p_{22})_v + p_{21}c_{12} + p_{22}c_{22} + q_{21}a_{12} + q_{22}a_{22} \\
&\qquad + (c_{22})_u - c_{21}p_{12} - c_{22}p_{22} + d_{22} = 0,
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
&(q_{11})_v + p_{11}d_{11} + p_{12}d_{21} + q_{11}b_{11} + q_{12}b_{21} \\
&\qquad + (d_{11})_u - c_{11}q_{11} - c_{12}q_{21} = 0, \\
&(q_{12})_v + p_{11}d_{12} + p_{12}d_{22} + q_{11}b_{12} + q_{12}b_{22} \\
&\qquad + (d_{12})_u - c_{11}q_{12} - c_{12}q_{22} = 0, \\
&(q_{21})_v + p_{21}d_{11} + p_{22}d_{21} + q_{21}b_{11} + q_{22}b_{21} \\
&\qquad + (d_{21})_u - c_{21}q_{11} - c_{22}q_{21} = 0, \\
&(q_{22})_v + p_{21}d_{12} + p_{22}d_{22} + q_{21}b_{12} + q_{22}b_{22} \\
&\qquad + (d_{22})_u - c_{21}q_{12} - c_{22}q_{22} = 0.
\end{aligned} \tag{10}
$$

If these conditions are satisfied all of the partial derivatives of higher order, of either y or z, will be determined by unique expressions, linear and homogeneous in y, z, y_u, and z_u; the existence of analytic function solutions of (S) involving four arbitrary constants linearly and in homogeneous fashion follows at once, and the complete integrability of system (S) is established.

From the first and fourth of equations (9) we find by addition

$$(p_{11} + p_{22})_v + (c_{11} + c_{22} + b_{11} + b_{22})_u = 0. \tag{11}$$

Consequently we may write

$$p_{11} + p_{22} = p_u, \qquad b_{11} + b_{22} + c_{11} + c_{22} = -p_v, \tag{12}$$

where p is a function of u and v determined by (12) except for an additive constant.

3. The second one-parameter family of ruled surfaces determined by (S)

We have seen that a system of form (S) may be utilized for the purpose of studying the one-parameter family of ruled surfaces $v = \text{const.}$ But

clearly we obtain a second one-parameter family of ruled surfaces if we equate u to a constant. We may find the system (S') which is most convenient for the purpose of studying these surfaces, by simple eliminations from equations already deduced.

The first order equations of (S') may be obtained at once by solving (4) for y_u and z_u. If we substitute the values of y_u and z_u, obtained in this way, into the equations (6) for y_{vv} and z_{vv} we obtain the second order equations of system (S'). We find in this way;

$$(S') \qquad \begin{aligned} y_{vv} + r_{11}\,y_v + r_{12}\,z_v + s_{11}\,y + s_{12}\,z &= 0, \\ z_{vv} + r_{21}\,y_v + r_{22}\,z_v + s_{21}\,y + s_{22}\,z &= 0, \\ y_u = g_{11}\,y_v + g_{12}\,z_v + h_{11}\,y + h_{12}\,z, \\ z_u = g_{21}\,y_v + g_{22}\,z_v + h_{21}\,y + h_{22}\,z, \end{aligned}$$

where we are assuming that

$$a_2 = a_{11}\,a_{22} - a_{12}\,a_{21} \tag{13}$$

is different from zero, and where we have put

$$a_2\,g_{11} = a_{22}, \quad a_2\,g_{12} = -a_{12}, \quad a_2\,g_{21} = -a_{21}, \quad a_2\,g_{22} = a_{11},$$

$$a_2\,h_{11} = a_{12}\,b_{21} - a_{22}\,b_{11}, \quad a_2\,h_{12} = a_{12}\,b_{22} - a_{22}\,b_{12}, \tag{14}$$

$$a_2\,h_{21} = a_{21}\,b_{11} - a_{11}\,b_{21}, \quad a_2\,h_{22} = a_{21}\,b_{12} - a_{11}\,b_{22},$$

and

$$a_2\,r_{11} = -(e_{11}\,a_{22} - e_{12}\,a_{21}), \quad a_2\,r_{12} = -(a_{11}\,e_{12} - a_{12}\,e_{11}),$$

$$a_2\,r_{21} = -(e_{21}\,a_{22} - e_{22}\,a_{21}), \quad a_2\,r_{22} = -(a_{11}\,e_{22} - a_{12}\,e_{21}),$$

$$a_2\,s_{11} = -\begin{vmatrix} f_{11} & e_{11} & e_{12} \\ b_{11} & a_{11} & a_{12} \\ b_{21} & a_{21} & a_{22} \end{vmatrix}, \quad a_2\,s_{12} = -\begin{vmatrix} f_{12} & e_{11} & e_{12} \\ b_{12} & a_{11} & a_{12} \\ b_{22} & a_{21} & a_{22} \end{vmatrix}, \tag{15}$$

$$a_2\,s_{21} = -\begin{vmatrix} f_{21} & e_{21} & e_{22} \\ b_{11} & a_{11} & a_{12} \\ b_{21} & a_{21} & a_{22} \end{vmatrix}, \quad a_2\,s_{22} = -\begin{vmatrix} f_{22} & e_{21} & e_{22} \\ b_{12} & a_{11} & a_{12} \\ b_{22} & a_{21} & a_{22} \end{vmatrix}.$$

If $a_2 = 0$, the two first order equations of (S') are replaced by

$$a_{22}\,y_v - a_{12}\,z_v + (a_{12}\,b_{21} - a_{22}\,b_{11})\,y + (a_{12}\,b_{22} - a_{22}\,b_{12})\,z = 0, \tag{16}$$

which shows that, in this case, the ruled surfaces u = const. are developable. This gives us the theorem: *the ruled surfaces u = const. are developables if and only if*

$$a_2 = a_{11}\,a_{22} - a_{12}\,a_{21} = 0.$$

The system of ruled surfaces, composed of the two one-parameter families $u = \text{const.}$ and $v = \text{const.}$, shall be called a *net of ruled surfaces.*

4. The differential equations of the surfaces of reference

If we return to equations (1), we observe that P_y and P_z describe, in general, two surfaces S_y and S_z when u and v vary over their ranges. In fact we may regard these surfaces and a general point-to-point correspondence between them as being given in advance, for the purpose of defining our two one-parameter families of ruled surfaces, or the net composed of both of them. We shall speak of the surfaces S_y and S_z as the *surfaces of reference.*

The differential equations of the surface S_y are obtained from

$$
\begin{aligned}
a_{11}\,y_u - y_v + b_{11}\,y &= -a_{12}\,z_u - b_{12}\,z,\\
y_{uu} + p_{11}\,y_u + * + q_{11}\,y &= -p_{12}\,z_u - q_{12}\,z,\\
-y_{uv} + c_{11}\,y_u + * + d_{11}\,y &= -c_{12}\,z_u - d_{12}\,z,\\
-y_{vv} + e_{11}\,y_u + * + f_{11}\,y &= -e_{12}\,z_u - f_{12}\,z,
\end{aligned}
\tag{17}
$$

by elimination of z_u and z. The differential equations of S_z are obtained from

$$
\begin{aligned}
a_{22}\,z_u - z_v + b_{22}\,z &= -a_{21}\,y_u - b_{21}\,y,\\
z_{uu} + p_{22}\,z_u + * + q_{22}\,z &= -p_{21}\,y_u - q_{21}\,y,\\
-z_{uv} + c_{22}\,z_u + * + d_{22}\,z &= -c_{21}\,y_u - d_{21}\,y,\\
-z_{vv} + e_{22}\,z_u + * + f_{22}\,z &= -e_{21}\,y_u - f_{21}\,y,
\end{aligned}
\tag{18}
$$

by elimination of y_u and y.

From some very familiar theorems we deduce the following results.

The curves $v = \text{const.}$ are asymptotic curves on S_y, if and only if

$$
\begin{vmatrix} a_{12} & b_{12} \\ p_{12} & q_{12} \end{vmatrix} = 0. \tag{19a}
$$

They are asymptotic curves on S_z, if and only if

$$
\begin{vmatrix} a_{21} & b_{21} \\ p_{21} & q_{21} \end{vmatrix} = 0. \tag{19b}
$$

The curves $u = \text{const.}$ are asymptotic curves on S_y or S_z respectively according as the conditions

$$
\begin{vmatrix} a_{12} & b_{12} \\ e_{12} & f_{12} \end{vmatrix} = 0 \quad \text{or} \tag{20a}
$$

$$
\begin{vmatrix} a_{21} & b_{21} \\ e_{21} & f_{21} \end{vmatrix} = 0 \tag{20b}
$$

are satisfied.

The curves $u = \text{const.}$ and $v = \text{const.}$ form a conjugate system on S_y, if and only if

$$\begin{vmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{vmatrix} = 0; \tag{21a}$$

the corresponding condition for S_z is

$$\begin{vmatrix} a_{21} & b_{21} \\ c_{21} & d_{21} \end{vmatrix} = 0. \tag{21b}$$

5. The seminvariants

Let us transform system (S) by a transformation of the form

$$y = \alpha\bar{y} + \beta\bar{z}, \qquad z = \gamma\bar{y} + \delta\bar{z}, \tag{22}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary functions of u and v, for which

$$\Delta = \alpha\delta - \beta\gamma \neq 0. \tag{23}$$

The result will be a new system of form (S) whose coefficients, $\bar{p}_{ik}$, $\bar{q}_{ik}$, $\bar{a}_{ik}$, and $\bar{b}_{ik}$ depend upon $\alpha, \beta, \gamma, \delta$ and upon the values of p_{ik}, q_{ik}, a_{ik}, and b_{ik}. Geometrically, this transformation has the effect of replacing the surfaces of reference, S_y and S_z, by two other surfaces, S^- and $S_{\bar{z}}$. The functions of p_{ik}, q_{ik}, a_{ik}, and b_{ik} which are left unchanged by this transformation, the *seminvariants*, are quantities whose values are independent of the choice of these surfaces of reference.

The effect of transformations of the form (22) upon the coefficients of (S) is given by the following equations which are important for what follows. We find

$$\begin{aligned}
\Delta\bar{p}_{11} &= 2(\alpha_u\delta - \gamma_u\beta) + p_{11}\alpha\delta + p_{12}\gamma\delta - p_{21}\alpha\beta - p_{22}\beta\gamma, \\
\Delta\bar{p}_{12} &= 2(\beta_u\delta - \delta_u\beta) + p_{11}\beta\delta + p_{12}\delta^2 - p_{21}\beta^2 - p_{22}\beta\delta, \\
\Delta\bar{p}_{21} &= -2(\alpha_u\gamma - \gamma_u\alpha) - p_{11}\alpha\gamma - p_{12}\gamma^2 + p_{21}\alpha^2 + p_{22}\alpha\gamma, \\
\Delta\bar{p}_{22} &= -2(\beta_u\gamma - \delta_u\alpha) - p_{11}\beta\gamma - p_{12}\gamma\delta + p_{21}\alpha\beta + p_{22}\alpha\delta,
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
\Delta\bar{q}_{11} &= \alpha_{uu}\delta - \gamma_{uu}\beta + p_{11}\alpha_u\delta + p_{12}\gamma_u\delta - p_{21}\alpha_u\beta - p_{22}\gamma_u\beta \\
&\qquad + q_{11}\alpha\delta + q_{12}\gamma\delta - q_{21}\alpha\beta - q_{22}\beta\gamma, \\
\Delta\bar{q}_{12} &= \beta_{uu}\delta - \delta_{uu}\beta + p_{11}\beta_u\delta + p_{12}\delta_u\delta - p_{21}\beta_u\beta - p_{22}\delta_u\beta \\
&\qquad + q_{11}\beta\delta + q_{12}\delta^2 - q_{21}\beta^2 - q_{22}\beta\delta, \\
\Delta\bar{q}_{21} &= -(\alpha_{uu}\gamma - \gamma_{uu}\alpha) - p_{11}\alpha_u\gamma - p_{12}\gamma_u\gamma + p_{21}\alpha_u\alpha + p_{22}\gamma_u\alpha \\
&\qquad - q_{11}\alpha\gamma - q_{12}\gamma^2 + q_{21}\alpha^2 + q_{22}\alpha\gamma, \\
\Delta\bar{q}_{22} &= -(\beta_{uu}\gamma - \delta_{uu}\alpha) - p_{11}\beta_u\gamma - p_{12}\delta_u\gamma + p_{21}\beta_u\alpha + p_{22}\delta_u\alpha \\
&\qquad - q_{11}\beta\gamma - q_{12}\gamma\delta + q_{21}\alpha\beta + q_{22}\alpha\delta.
\end{aligned} \tag{25}$$

either by direct calculation or else from the theory of ruled surfaces.* We find further

$$
\begin{aligned}
\Delta\bar{a}_{11} &= \alpha\delta a_{11} + \gamma\delta a_{12} - \alpha\beta a_{21} - \beta\gamma a_{22}, \\
\Delta\bar{a}_{12} &= \beta\delta a_{11} + \delta^2 a_{12} - \beta^2 a_{21} - \beta\delta a_{22}, \\
\Delta\bar{a}_{21} &= -\alpha\gamma a_{11} - \gamma^2 a_{12} + \alpha^2 a_{21} + \alpha\gamma a_{22}, \\
\Delta\bar{a}_{22} &= -\beta\gamma a_{11} - \gamma\delta a_{12} + \alpha\beta a_{21} + \alpha\delta a_{22},
\end{aligned} \tag{26}
$$

and

$$
\begin{aligned}
\Delta\bar{b}_{11} &= -\alpha_v\,\delta + \gamma_v\,\beta + \alpha_u\,\delta a_{11} + \gamma_u\,\delta a_{12} - \alpha_u\,\beta a_{21} - \gamma_u\,\beta a_{22} \\
&\qquad + \alpha\delta b_{11} + \gamma\delta b_{12} - \alpha\beta b_{21} - \beta\gamma b_{22}, \\
\Delta\bar{b}_{12} &= -\beta_v\,\delta + \delta_v\,\beta + \beta_u\,\delta a_{11} + \delta_u\,\delta a_{12} - \beta_u\,\beta a_{21} - \delta_u\,\beta a_{22} \\
&\qquad + \beta\delta b_{11} + \delta^2 b_{12} - \beta^2 b_{21} - \beta\delta b_{22}, \\
\Delta\bar{b}_{21} &= \alpha_v\,\gamma - \gamma_v\,\alpha - \alpha_u\,\gamma a_{11} - \gamma_u\,\gamma a_{12} + \alpha_u\,\alpha a_{21} + \gamma_u\,\alpha a_{22} \\
&\qquad - \alpha\gamma b_{11} - \gamma^2 b_{12} + \alpha^2 b_{21} + \alpha\gamma b_{22}, \\
\Delta\bar{b}_{22} &= \beta_v\,\gamma - \delta_v\,\alpha - \beta_u\,\gamma a_{11} - \delta_u\,\gamma a_{12} + \beta_u\,\alpha a_{21} + \delta_u\,\alpha a_{22} \\
&\qquad - \beta\gamma b_{11} - \delta\gamma b_{12} + \alpha\beta b_{21} + \alpha\delta b_{22}.
\end{aligned} \tag{27}
$$

From the theory of ruled surfaces we know that the quantities

$$
\begin{aligned}
u_{11} &= 2(p_{11})_u - 4q_{11} + p_{11}^2 + p_{12}\,p_{21}, \\
u_{12} &= 2(p_{12})_u - 4q_{12} + p_{12}(p_{11} + p_{22}), \\
u_{21} &= 2(p_{21})_u - 4q_{21} + p_{21}(p_{11} + p_{22}), \\
u_{22} &= 2(p_{22})_u - 4q_{22} + p_{22}^2 + p_{12}\,p_{21},
\end{aligned} \tag{28}
$$

are transformed in accordance with the equations obtained from (26) if we replace a_{ik} by u_{ik}, and $\bar{a}_{ik}$ by $\bar{u}_{ik}$.† We express this fact by saying that *the a_{ik}'s and u_{ik}'s are cogredient* for transformations of the form (22).

From the u_{ik}'s and p_{ik}'s we form a new set of four quantities

$$
\begin{aligned}
u_{11}^{(1)} &= v_{11} = 2(u_{11})_u + p_{12}\,u_{21} - p_{21}\,u_{12}, \\
u_{12}^{(1)} &= v_{12} = 2(u_{12})_u + (p_{11} - p_{22})\,u_{12} - p_{12}(u_{11} - u_{22}), \\
u_{21}^{(1)} &= v_{21} = 2(u_{21})_u - (p_{11} - p_{22})\,u_{21} + p_{21}(u_{11} - u_{22}), \\
u_{22}^{(1)} &= v_{22} = 2(u_{22})_u - p_{12}\,u_{21} + p_{21}\,u_{12},
\end{aligned} \tag{29}
$$

which are also cogredient with the u_{ik}'s and therefore with the a_{ik}'s.‡ The

* *Proj. Diff. Geom.*, pages 102–103.

† Ibid., pages 96 and 103.

‡ Ibid., pp. 99 and 100.

notation $u_{ik}^{(1)}$ *indicates a process,* defined by equations (29), by means of which from a set of four quantities, u_{ik}, may be derived four new quantities, $u_{ik}^{(1)}$, cogredient with the former set. Clearly this process may be repeated, giving rise to four new quantities

$$u_{ik}^{(11)} = v_{ik}^{(1)} = w_{ik}, \tag{30}$$

also cogredient with the u_{ik}'s. Any further application of this process to the u_{ik}'s is unnecessary, since the new variables obtained in this way are linear combinations of the u_{ik}'s, v_{ik}'s, and w_{ik}'s with seminvariant coefficients.†

The variables u_{ik}, v_{ik}, and w_{ik}, are fundamental in the theory of ruled surfaces, but the variables a_{ik} do not occur in that theory at all. Since the a_{ik}'s are cogredient with the u_{ik}'s we can now obtain further sets of cogredient variables by applying the process defined by (29) to the a_{ik}'s. Thus we obtain the new sets of cogredient quantities $a_{ik}^{(1)}$, $a_{ik}^{(11)}$, etc.

From the theory of ruled surfaces we know that

$$\begin{aligned} I &= u_{11} + u_{22}, & J &= u_{11}\,u_{22} - u_{12}\,u_{21}, \\ K &= v_{11}\,v_{22} - v_{12}\,v_{21}, & L &= w_{11}\,w_{22} - w_{12}\,w_{21}, \end{aligned}$$

$$\Delta = \begin{vmatrix} u_{11} - u_{22}, & v_{11} - v_{22}, & w_{11} - w_{22} \\ u_{12} & v_{12} & w_{12} \\ u_{21} & v_{21} & w_{21} \end{vmatrix} \tag{31}$$

are seminvariants,‡ a fact which may moreover be verified directly from equations (29) and the cogredience properties noted. Of course $v_{11} + v_{22}$ and $w_{11} + w_{22}$ are also seminvariants; we have not listed them since the relations

$$v_{11} + v_{22} = 2I_u, \qquad w_{11} + w_{22} = 4I_{uu}$$

enable us to express them as derivatives of I. Moreover Δ is really not independent of I, J, K, L and of their derivatives, but it is so important a combination as to merit special mention. All of the seminvariants of a single ruled surface are functions of I, J, K, L and of their derivatives with respect to u.§

From the cogredience properties which we have noted, it follows at once that the quantities

$$\begin{aligned} a_1 &= a_{11} + a_{22}, & a_2 &= a_{11}\,a_{22} - a_{12}\,a_{21}, \\ 2(a_1)_u &= a_{11}^{(1)} + a_{22}^{(1)}, & a_2^{(1)} &= a_{11}^{(1)}\,a_{22}^{(1)} - a_{12}^{(1)}\,a_{21}^{(1)}, \qquad \text{etc.} \end{aligned} \tag{32}$$

are also seminvariants of (S). Moreover, if λ denotes an arbitrary constant,

* *Proj. Diff. Geom.*, p. 101.
† Ibid., p. 101.
‡ Ibid., pp. 97 to 102
§ Ibid., p. 101.

$$\begin{vmatrix} a_{11} + \lambda u_{11}, & a_{12} + \lambda u_{12} \\ a_{21} + \lambda u_{21}, & a_{22} + \lambda u_{22} \end{vmatrix}$$

will be a seminvariant for all values of λ. Consequently we find new seminvariants of the following type:

$$\begin{aligned} (a, u) &= a_{11} u_{22} + a_{22} u_{11} - a_{12} u_{21} - a_{21} u_{12}, \\ (a, v) &= a_{11} v_{22} + a_{22} v_{11} - a_{12} v_{21} - a_{21} v_{12} = (a, u^{(1)}), \qquad \text{etc.} \end{aligned} \tag{33}$$

Of course, many of these are expressible in other forms. Thus we have, for instance,

$$(u, v) = (u, u^{(1)}) = 2J_u, \qquad (a, a^{(1)}) = 2(a_2)_u. \tag{33a}$$

The seminvariant Δ is a very important representative of a type of seminvariants expressible as third order determinants formed from three distinct cogredient sets, and may be represented by the symbol

$$\Delta = (u, v, w).$$

Clearly, we can form other seminvariants of this form, such as

$$(a, u, v), \qquad (a^{(1)}, u, v), \qquad (a, a^{(1)}, u), \qquad \text{etc.}. \tag{34}$$

All of the seminvariants obtained so far contain only the coefficients p_{ik}, q_{ik}, and a_{ik}, of system (S), besides partial derivatives of these quantities with respect to u.

In order to find seminvariants which involve the quantities b_{ik} also, and partial derivatives with respect to v, we might operate in exactly similar fashion upon the coefficients r_{ik}, s_{ik}, g_{ik}, h_{ik} of the system (S') of Art. 3. But we shall show how to obtain simpler seminvariants of the desired kind by a different method.

For this purpose we first re-write system (S) in a different form. We put

$$\rho = 2y_u + p_{11} y + p_{12} z, \qquad \sigma = 2z_u + p_{21} y + p_{22} z. \tag{35}$$

The points, P_ρ and P_σ, whose homogeneous coördinates are given by these expressions are such that the line $P_\rho P_\sigma$ is a generator of the same set as $P_y P_z$ on the quadric surface H which osculates the ruled surface v = const. along $P_y P_z$, while $P_y P_\rho$ and $P_z P_\sigma$ are two generators of the second set of H.*

As a result of (35), the equations of (S) assume the form

$$\begin{aligned} 2\rho_u + p_{11}\rho + p_{12}\sigma &= u_{11} y + u_{12} z, \\ 2\sigma_u + p_{21}\rho + p_{22}\sigma &= u_{21} y + u_{22} z, \dagger \\ 2y_v + \pi_{11} y + \pi_{12} z &= a_{11}\rho + a_{12}\sigma, \\ 2z_v + \pi_{21} y + \pi_{22} z &= a_{21}\rho + a_{22}\sigma, \end{aligned} \tag{36}$$

* *Proj. Diff. Geom.*, p. 147.

† Ibid., p. 148.

where

$$\pi_{11} = a_{11}\,p_{11} + a_{12}\,p_{21} - 2b_{11}, \qquad \pi_{12} = a_{11}\,p_{12} + a_{12}\,p_{22} - 2b_{12},$$
$$\pi_{21} = a_{21}\,p_{11} + a_{22}\,p_{21} - 2b_{21}, \qquad \pi_{22} = a_{21}\,p_{12} + a_{22}\,p_{22} - 2b_{22}. \tag{37}$$

But corresponding to all transformations of the form (22), the variables, ρ and σ, defined by (35) undergo the cogredient transformations

$$\rho = \alpha\bar{\rho} + \beta\bar{\sigma}, \qquad \sigma = \gamma\bar{\rho} + \delta\bar{\sigma}.*$$

Consequently it follows, by comparing the first two equations of (36) with the last two, and making use of the various cogredience properties already noted; that the quantities π_{ik}, introduced by (37), must be cogredient with the quantities p_{ik} except that the partial derivatives of $\alpha, \beta, \gamma, \delta$ with respect to u which occur in (24) must be replaced by v-derivatives.

From this remark we conclude further that, from a set of quantities like a_{ik}, or any cogredient set, we can obtain a new set of cogredient variables if, in the process which is exemplified by (29), we replace the quantities p_{ik} by π_{ik} and at the same time replace the u-derivatives by v-derivatives. We use an upper index 2 to indicate this new process. Thus, for instance, we obtain

$$a_{11}^{(2)} = 2\,(a_{11})_v + \pi_{12}\,a_{21} - \pi_{21}\,a_{12},$$
$$a_{12}^{(2)} = 2\,(a_{12})_v + (\pi_{11} - \pi_{22})\,a_{12} - \pi_{12}\,(a_{11} - a_{22}), \qquad \text{etc.} \tag{38}$$

From the quantities obtained in this way we form seminvariants as before. Thus

$$a_{11}^{(2)} + a_{22}^{(2)} = 2\,(a_{11} + a_{22})_v, \qquad a_{11}^{(2)}\,a_{22}^{(2)} - a_{12}^{(2)}\,a_{21}^{(2)} = a_2^{(2)},$$
$$a_{11}^{(22)}\,a_{22}^{(22)} - a_{12}^{(22)}\,a_{22}^{(22)} = a_2^{(22)}, \qquad u_{11}^{(2)}\,u_{22}^{(2)} - u_{12}^{(2)}\,u_{21}^{(2)} = u_2^{(2)}, \qquad \text{etc.} \tag{39}$$

are seminvariants. Moreover the two processes may be combined. Thus the quantities $(a_{ik}^{(1)})^{(2)}$ are cogredient with the quantities a_{ik}, and give rise to a seminvariant

$$(a_{11}^{(1)})^{(2)}\,(a_{22}^{(1)})^{(2)} - (a_{12}^{(1)})^{(2)}\,(a_{21}^{(1)})^{(2)} = a_2^{(12)}; \tag{40}$$

the similar formation which results when the two processes are used in opposite order may be denoted by $a_2^{(21)}$.

We have noticed already that, from two or three sets of quantities cogredient with the a_{ik}'s, bilinear or trilinear seminvariants of type (33) or (34) may be formed. By applying this remark to the new variables just obtained we find such new seminvariants as

$$(a^{(2)}, u), \qquad (a, u^{(2)}), \qquad \text{etc.},$$
$$(a, a^{(1)}, a^{(2)}), \qquad (a^{(1)}, u, v), \qquad (a^{(2)}, u, v), \qquad \text{etc.}$$

* *Proj. Diff. Geom.*, p. 146.

6. The flecnodal canonical form

Let us assume that

$$\theta_4 = (u_{11} - u_{22})^2 + 4u_{12}\,u_{21}, \tag{41}$$

is not equal to zero. Then the flecnode curve on each of the ruled surfaces $v = \text{const.}$ has two distinct branches. The locus of these curves is in general a surface of two distinct sheets which we shall call the *locus of the flecnodes* of our one-parameter family of ruled surfaces. We use this locution to avoid confusion with the *flecnode surface*, which is a ruled surface the locus of the flecnode tangents of a single ruled surface. In general, the totality of the flecnode tangents of a one-parameter family of ruled surfaces will be a congruence of lines; the totality of the flecnode surfaces of a one-parameter family of ruled surfaces will be a new one-parameter family of ruled surfaces. We shall speak of them as the *congruence of flecnode tangents*, and the *one-parameter family of flecnode surfaces* respectively. Again the congruence of flecnode tangents should not be confused with what we have formerly called the flecnode congruence of a single ruled surface. The locus of these for a one-parameter family of ruled surfaces gives rise, in general to a complex, the *flecnode complex* of the family.*

If the two sheets of the locus of flecnodes are distinct, that is, if $\theta_4 \neq 0$, we may use them as surfaces of reference for the system (S). We shall then have

$$u_{12} = u_{21} = 0, \qquad u_{11} - u_{22} \neq 0.\dagger \tag{42}$$

If these conditions are satisfied, they will still be satisfied after any transformation of the form

$$y = \alpha\bar{y}, \qquad z = \delta\bar{z},$$

and α and δ may moreover be chosen, according to (24), in such a way as to make $\bar{p}_{11} = \bar{p}_{22} = 0$. Thus, if $\theta_4 \neq 0$, we may suppose

$$u_{12} = u_{21} = 0, \qquad u_{11} - u_{22} \neq 0, \qquad p_{11} = p_{22} = 0. \tag{43}$$

When the coefficients of (S) satisfy these conditions and one other condition to be formulated presently we shall say that (S) is in the *flecnodal canonical form.*

We propose to show that *all of the coefficients of the flecnodal canonical form are expressible in terms of seminvariants of the original system* (S).

In fact we have,‡ under the assumption $\theta_4 \neq 0$, $\theta_{10} \neq 0$,

* *Proj. Diff. Geom.*, pp. 146–153, and pp. 175–190.

† Ibid., pp. 149–150.

‡ Ibid., p. 120, equ. (102).

$$p_{11} = 0, \qquad p_{22} = 0, \qquad q_{12} = \tfrac{1}{2}(p_{12})_u, \qquad q_{21} = \tfrac{1}{2}(p_{21})_u,$$

$$p_{12} = \frac{\sqrt{\theta_{10}}}{\theta_4} e^{\frac{1}{4}\int \sqrt{\theta_4}\frac{\theta_9}{\theta_{10}}du}, \qquad p_{21} = \frac{\sqrt{\theta_{10}}}{\theta_4} e^{-\frac{1}{4}\int \sqrt{\theta_4}\frac{\theta_9}{\theta_{10}}du},$$

$$64q_{11} = \frac{1}{\theta_4^2}[16\theta_{10} - \vartheta_{10} + 8\theta_4\theta_4'' - 9(\theta_4')^2] - 8\sqrt{\theta_4}, \tag{44}$$

$$64q_{22} = \frac{1}{\theta_4^2}[16\theta_{10} - \vartheta_{10} + 8\theta_4\theta_4'' - 9(\theta_4')^2] + 8\sqrt{\theta_4},$$

where θ_4 has already been defined, where θ_9 is the same as the quantity Δ defined by (31), and where

$$\begin{aligned}\theta_{10} &= (I^2 - 4J)(K - I_u)^2 + (II_u - 2J_u)^2 \\ &= \tfrac{1}{4}\{(u_{12}v_{21} - u_{21}v_{12})^2 - 4[(u_{11} - u_{22})v_{12} \\ &\qquad - (v_{11} - v_{22})u_{12}][(u_{11} - u_{22})v_{21} - (v_{11} - v_{22})u_{21}]\}, \\ \vartheta_{10} &= 8\theta_4(\theta_4)_{uu} - 9(\theta_4)_u^2 + 8I\theta_4^2.\end{aligned} \tag{45}$$

These quantities θ_4, θ_9, θ_{10}, ϑ_{10} are, in fact, not merely seminvariants of (S); they are invariants of the one-parameter family of ruled surfaces $v = \text{const.}$; ϑ_{10} in particular is the so-called quadri-derivative of θ_4 and is usually denoted by $\theta_{4\cdot 1}$, a notation which we shall have to abandon to avoid confusion in our later developments. Since the eight coefficients, p_{ik} and q_{ik}, of the flecnodal canonical form are now expressed in terms of θ_4, θ_9, θ_{10}, and ϑ_{10}, and since these quantities are functions of I, J, K, L, and of their partial derivatives with respect to u, it only remains to show that the remaining eight coefficients, a_{ik} and b_{ik}, of the flecnodal canonical form can also be expressed in terms of seminvariants. For this purpose we consider the seminvariants

$$a_1 = a_{11} + a_{22}, \qquad a_2 = a_{11}a_{22} - a_{12}a_{21};$$

$$(a, u) = a_{11}u_{22} + a_{22}u_{11} - a_{12}u_{21} - a_{21}u_{12},$$

$$(a, u, v) = \begin{vmatrix} a_{11} - a_{22}, & u_{11} - u_{22}, & v_{11} - v_{22} \\ a_{12} & u_{12} & v_{12} \\ a_{21} & u_{21} & v_{21} \end{vmatrix}. \tag{46}$$

In the canonical form, we have $u_{12} = u_{21} = 0$, and owing to the assumptions $\theta_4 \neq 0$, $\theta_{10} \neq 0$, the quantities $u_{11} - u_{22}$, v_{12}, and v_{21} will not be zero. Consequently equations (46) enable us to express the four coefficients a_{ik} of the canonical form in terms of a_1, a_2, (a, u), (a, u, v) and I, J, K, L and their derivatives.

Let us consider next the seminvariants

$$a_1^{(2)} = a_{11}^{(2)} + a_{22}^{(2)}, \qquad a_2^{(2)} = a_{11}^{(2)} a_{22}^{(2)} - a_{12}^{(2)} a_{21}^{(2)},$$

$$(a^{(2)}, u) = a_{11}^{(2)} u_{22} + a_{22}^{(2)} u_{11} - a_{12}^{(2)} u_{21} - a_{21}^{(2)} u_{12},$$

$$(47) \qquad (a, a^{(1)}, a^{(2)}) = \begin{vmatrix} a_{11} - a_{22}, & a_{11}^{(1)} - a_{22}^{(1)}, & a_{11}^{(2)} - a_{22}^{(2)} \\ a_{12}, & a_{12}^{(1)}, & a_{12}^{(2)} \\ a_{21}, & a_{21}^{(1)}, & a_{21}^{(2)} \end{vmatrix},$$

the first of which, $a_1^{(2)}$, need not be mentioned explicitly, since it is equal to $2(a_1)_v$. If the cofactors of $a_{12}^{(2)}$ and $a_{21}^{(2)}$ in $(a, a^{(1)}, a^{(2)})$ are not both equal to zero, these equations enable us to express the values of $a_{ik}^{(2)}$, for the canonical form, in terms of the seminvariants mentioned previously and of $a_2^{(2)}$, $(a^{(2)}, u)$, and $(a, a^{(1)}, a^{(2)})$. According to (38), we obtain in this way seminvariant expressions for π_{12}, π_{21}, and $\pi_{11} - \pi_{22}$. The integrability condition (11) may be written

$$(48) \qquad (\pi_{11} + \pi_{22})_u - (a_{11} + a_{22})_{uu} = (p_{11} + p_{22})_v.$$

In our canonical form, we have $p_{11} = p_{22} = 0$, so that (48) enables us to conclude that

$$\pi_{11} + \pi_{22} - (a_{11} + a_{22})_u = V(v)$$

is a function of v alone. But any transformation of the form

$$y = \alpha(v)\bar{y}, \qquad z = \delta(v)\bar{z},$$

where α and δ are functions of v alone, preserves the conditions $p_{11} = p_{22} = u_{12} = u_{21} = 0$, and transforms $\pi_{11} + \pi_{22} - (a_{11} + a_{22})_u$ into

$$\overline{\pi_{11} + \pi_{22} - (a_{11} + a_{22})_u} = \pi_{11} + \pi_{22} - (a_{11} + a_{22})_u + 2\left(\frac{\alpha_v}{\alpha} + \frac{\delta_v}{\delta}\right).$$

Consequently, by choosing $\alpha(v)$ and $\delta(v)$ in accordance with the condition

$$\frac{\alpha_v}{\alpha} + \frac{\delta_v}{\delta} = -\tfrac{1}{2}[\pi_{11} + \pi_{22} - (a_{11} + a_{22})_u] = -\tfrac{1}{2}V(v),$$

we may make $\bar{\pi}_{11} + \bar{\pi}_{22} - (\bar{a}_{11} + \bar{a}_{22})_u$ equal to zero. Let us assume that this transformation has been made, so that

$$(48a) \qquad \pi_{11} + \pi_{22} = (a_{11} + a_{22})_u;$$

this is the condition mentioned above, which together with

$$u_{12} = u_{21} = p_{11} = p_{22} = 0$$

characterizes the flecnodal canonical form of system (S).

By means of (48a), $\pi_{11} + \pi_{22}$ is expressed in terms of seminvariants; since we have shown before that π_{12}, π_{21}, and $\pi_{11} - \pi_{22}$ may be expressed in terms of certain seminvariants, it follows that all of the π_{ik}'s can be thus expressed. Finally, equations (37) show that the same thing is true of the b_{ik}'s.

We have obtained the following result. *If θ_4 and θ_{10} are different from zero, and if the co-factors of $a_{12}^{(2)}$ and $a_{21}^{(2)}$ in $(a, a^{(1)}, a^{(2)})$ are not both zero, all of the coefficients of the flecnodal canonical form are functions of the eleven seminvariants*

$$I, J, K, L, a_1, a_2, (a, u), (a, u, v), a_2^{(2)}, (a^{(2)}, u), (a, a^{(1)}, a^{(2)}),$$

and of partial derivatives of these seminvariants.

From this theorem it follows at once that *any seminvariant of (S) can be expressed as a function of the eleven seminvariants mentioned and of their partial derivatives, provided that θ_4 and θ_{10} do not vanish, and that at least one of the cofactors of $a_{12}^{(2)}$ or $a_{21}^{(2)}$ in $(a, a^{(1)}, a^{(2)})$ is not equal to zero.*

It is known from the theory of ruled surfaces that the cases $\theta_4 = 0$ and $\theta_{10} = 0$ are really exceptional; that is, in these cases the coefficients of the canonical form can not be expressed entirely in terms of the seminvariants mentioned. The additional distinction, however, as to whether the cofactors of $a_{12}^{(2)}$ and $a_{21}^{(2)}$ in $(a, a^{(1)}, a^{(2)})$ do or do not vanish, might be avoided by substituting for $(a, a^{(1)}, a^{(2)})$ some other seminvariant, such as $(a^{(2)}, v)$ for instance. We prefer, however, to retain $(a, a^{(1)}, a^{(2)})$ for reasons which will become apparent later.

7. Invariants of the net

The seminvariants are those combinations formed from coefficients of (S) whose values are independent of the choice of the surfaces of reference, S_y and S_z. But the value of a seminvariant is not, in general, independent of the parametric representation of the ruled surfaces of the net. If we make a transformation of the form

$$\bar{u} = U(u), \qquad \bar{v} = V(v), \tag{49}$$

where $U(u)$ and $V(v)$ are arbitrary functions of the single variables indicated, the parametric ruled surfaces of the net are left unchanged, but their parametric representation is altered. Those functions of seminvariants which are not changed at all by any transformation of form (49) shall be called *absolute invariants of the net.* A seminvariant $\theta_{k,l}$, which is transformed in accordance with the equation

$$\bar{\theta}_{k,l} = (U')^{-k} (V')^{-l} \theta_{k,l}$$

shall be called a *relative net invariant of weights k and l.*

The theory of ruled surfaces supplies us with a number of such net-invariants. The fundamental invariants of this sort are those denoted by θ_4, θ_9, θ_{10}, and ϑ_{10}, in Article 6. Moreover the theory of ruled surfaces enables us to form all net-invariants which depend merely on the coefficients p_{ik} and q_{ik}. They are obtained from the four fundamental ones by certain simple differentiation processes.*

In accordance with the general notation $\theta_{k,l}$ just established, the four fundamental net-invariants θ_4, θ_9, θ_{10}, ϑ_{10} shall now be denoted by $\theta_{4,0}$, $\theta_{9,0}$, $\theta_{10,0}$, and $\vartheta_{10,0}$ respectively.

In order to be in a position to find the remaining net-invariants, we study the effect of a transformation of form (49) upon the coefficients of (S). We find the following transformation equations;

$$\bar{p}_{11} = \frac{1}{U'}(p_{11} + \eta), \qquad \bar{p}_{12} = \frac{p_{12}}{U'}, \qquad \bar{p}_{21} = \frac{p_{21}}{U'}, \qquad \bar{p}_{22} = \frac{1}{U'}(p_{22} + \eta),$$

$$\bar{q}_{ik} = \frac{q_{ik}}{(U')^2}, \qquad \bar{a}_{ik} = \frac{U'}{V'}a_{ik}, \qquad \bar{b}_{ik} = \frac{b_{ik}}{V'}, \tag{50}$$

where

$$\eta = \frac{U''}{U'}. \tag{51}$$

From these equations, the following may be deduced;

$$\bar{u}_{11} = \frac{1}{(U')^2}(u_{11} + 2\mu), \qquad \bar{u}_{12} = \frac{1}{(U')^2}u_{12},$$

$$\bar{u}_{21} = \frac{1}{(U')^2}u_{21}, \qquad \bar{u}_{22} = \frac{1}{(U')^2}(u_{22} + 2\mu),\dagger \tag{52}$$

where

$$\mu = \eta' - \tfrac{1}{2}\eta^2 = \frac{U'''}{U'} - \frac{3}{2}\left(\frac{U''}{U'}\right)^2 = \{U, u\} \tag{53}$$

is the Schwarzian derivative of U with respect to u. We find further

$$\bar{I} = \frac{1}{(U')^2}(I + 4\mu), \qquad \bar{J} = \frac{1}{(U')^4}(J + 2\mu I + 4\mu^2), \tag{54}$$

and

$$\bar{r}_{11} = \frac{1}{(U')^3}(r_{11} - 4u_{11}\eta + 4\mu' - 8\mu\eta), \qquad \bar{r}_{12} = \frac{1}{(U')^3}(r_{12} - 4u_{12}\eta),$$

$$\bar{r}_{21} = \frac{1}{(U')^3}(r_{21} - 4u_{21}\eta), \qquad \bar{r}_{22} = \frac{1}{(U')^3}(r_{22} - 4u_{22}\eta + 4\mu' - 8\mu\eta), \tag{55}$$

where μ' is the derivative of μ.

* *Proj. Diff. Geom.*, p. 121.

† Compare *Proj. Diff. Geom.*, p. 104.

The invariance of $\theta_{4,0}$, $\theta_{9,0}$, $\theta_{10,0}$, and $\vartheta_{10,0}$ has been established already. To obtain the remaining fundamental net-invariants, we must find the effect of these transformations upon the seven seminvariants a_1, a_2, (a, u), (a, u, v), $a_2^{(2)}$, $(a^{(2)}, u)$, and $(a, a^{(1)}, a^{(2)})$. From (46) and (50) we see that

$$\bar{a}_1 = \frac{U'}{V'} a_1, \qquad \bar{a}_2 = \left(\frac{U'}{V'}\right)^2 a_2. \tag{56}$$

Therefore a_1 and a_2 are relative net-invariants which may be denoted by

$$\theta_{-1,1} = a_1, \qquad \theta_{-2,2} = a_2.$$

Of course

$$A = a_1^2 - 4a_2 = (a_{11} - a_{22})^2 + 4a_{12}a_{21} \tag{57}$$

is also a relative net-invariant. The special importance of A will appear later.

We find further

$$(\overline{a, u}) = \frac{1}{U'V'}[(a, u) + 2\mu a_1]. \tag{58}$$

Combining this with (54) and (56), we see that

$$\theta_{1,1} = (a, u) - \tfrac{1}{2} I a_1 = -a_{12}u_{21} - a_{21}u_{12} - \tfrac{1}{2}(a_{11} - a_{22})(u_{11} - u_{22}) \tag{59}$$

is a new net-invariant. The seminvariant (a, u, v) is transformed in accordance with the equation

$$(\overline{a, u, v}) = \frac{1}{(U')^4 V'}(a, u, v),$$

so that

$$\theta_{4,1} = (a, u, v) \tag{60}$$

is a net-invariant.

We find further

$$\bar{\pi}_{ik} = \frac{1}{V'}(\pi_{ik} + \eta a_{ik}),$$

$$(\bar{a}_{ik})_{\bar{u}} = \frac{1}{V'}[(a_{ik})_u + \eta a_{ik}], \qquad (\bar{a}_{ik})_{\bar{v}} = \frac{U'}{(V')^2}[(a_{ik})_u - \zeta a_{ik}], \tag{61}$$

where

$$\zeta = \frac{V''}{V'}, \tag{62}$$

so that

$$\bar{a}_{ik}^{(1)} = \frac{1}{V'}(a_{ik}^{(1)} + 2\eta a_{ik}), \qquad \bar{a}_{ik}^{(2)} = \frac{U'}{(V')^2}(a_{ik}^{(2)} - 2\zeta a_{ik}). \tag{63}$$

Consequently we find

$$\bar{a}_2^{(2)} = \frac{(U')^2}{(V')^4}[a_2^{(2)} - 4\zeta(a_2)_v + 4\zeta^2 a_2],$$

$$(64) \qquad (a^{(2)}, u) = \frac{1}{(U')(V')^2}[(a^{(2)}, u) - 2\zeta(a, u) + 4\mu(a_1)_v - 4\mu\zeta a_1].$$

$$(\bar{a}, \bar{a}^{(1)}, \bar{a}^{(2)}) = \frac{(U')^2}{(V')^4}(a, a^{(1)}, a^{(2)}).$$

Thus, the seminvariant

$$(65) \qquad \theta_{-2,4} = (a, a^{(1)}, a^{(2)})$$

is also a net-invariant, and it remains to find two further net-invariants, one involving $a_2^{(2)}$ and one involving $(a^{(2)}, u)$.

Let us introduce the seminvariant

$$(66) \qquad A^{(2)} = (a_1^{(2)})^2 - 4a_2^{(2)}$$

analogous to A. We find

$$(67) \qquad \bar{A}^{(2)} = \frac{(U')^2}{(V')^4}[A^{(2)} - 4\zeta A_v + 4\zeta^2 A].$$

Moreover we have

$$(68) \qquad (\bar{A})_{\bar{u}} = \frac{U'}{(V')^2}(A_u + 2\eta A), \qquad (\bar{A})_{\bar{v}} = \frac{(U')^2}{(V')^3}(A_v - 2\zeta A).$$

Consequently

$$(69) \qquad \begin{aligned}\theta_{-4,6} = AA^{(2)} - (A_v)^2 = (a_1^2 - 4a_2)[(a_1^{(2)})^2 - 4a_2^{(2)}] \\ - 4[a_1(a_1)_v - 2(a_2)_v]^2\end{aligned}$$

is a new net-invariant.

Finally we find, from (58),

$$(70) \qquad (\overline{a, u})_{\bar{v}} = \frac{1}{U'(V')^2}[(a, u)_v - \zeta(a, u) + 2\mu(a_1)_v - 2\mu\zeta a_1],$$

which proves that

$$(71) \qquad \theta_{1,2} = (a^{(2)}, u) - 2(a, u)_v = -(a, u^{(2)})$$

is a net-invariant.

From the results of Article 6 it now follows that all of the coefficients of the flecnodal canonical form can be expressed in terms of the net-invariants

$$\theta_{0,4}, \theta_{0,9}, \theta_{0,10}, \vartheta_{0,10}, a_1, a_2, \theta_{1,1}, \theta_{4,1}, \theta_{-4,6}, \theta_{1,2}, \theta_{-2,4}.$$

These eleven net-invariants may therefore be regarded as fundamental. In fact, if they are given as functions of u and v, the net of ruled surfaces is determined except for projective transformations, and all other net-invariants are expressible in terms of these eleven and others derived from them by certain differentiation processes. Of course the eleven invariants cannot be assigned arbitrarily. The relations which exist between them can be found by writing the integrability conditions for a system (S) in its flecnodal canonical form.

8. Invariants of the one-parameter family of ruled surfaces v = constant

If we transform our system (S) by a transformation of the form

$$\bar{u} = \phi(u, v), \qquad \bar{v} = V(v), \tag{72}$$

where $\phi(u, v)$ is an arbitrary function of both u and v, while $V(v)$ is an arbitrary function of v alone, the one-parameter family of ruled surfaces $\bar{v}$ = const. will coincide with the family v = const., but the family $\bar{u}$ = const. will not coincide with u = const. In fact the ruled surfaces $\bar{u}$ = const. may, by choice of $\phi(u, v)$, be made to coincide with any one-parameter family made up of the generators of the ruled surfaces v = const. Consequently, those combinations of net-invariants which remain invariant under all transformations of form (72) will be invariants of the one-parameter family of ruled surfaces v = const.

If we again use the notation $\bar{p}_{ik}$, $\bar{q}_{ik}$, etc., to denote the coefficients of the system of differential equations obtained from (S) by a transformation of form (72), we find

$$\begin{aligned}
\bar{p}_{11} &= \frac{1}{\phi_u}\left(p_{11} + \frac{\phi_{uu}}{\phi_u}\right), & \bar{p}_{12} &= \frac{p_{12}}{\phi_u}, \\
\bar{p}_{21} &= \frac{p_{21}}{\phi_u}, & \bar{p}_{22} &= \frac{1}{\phi_u}\left(p_{22} + \frac{\phi_{uu}}{\phi_u}\right), \\
\bar{q}_{ik} &= \frac{q_{ik}}{\phi_u^2}, & \bar{b}_{ik} &= \frac{b_{ik}}{V'}, \\
\bar{a}_{11} &= \frac{1}{V'}(a_{11}\phi_u - \phi_v), & \bar{a}_{12} &= \frac{\phi_u}{V'}a_{12}, \\
\bar{a}_{21} &= \frac{\phi_u}{V'}a_{21}, & \bar{a}_{22} &= \frac{1}{V'}(a_{22}\phi_u - \phi_v),
\end{aligned} \tag{73}$$

whence

$$\bar{u}_{11} = \frac{1}{\phi_u^2}(u_{11} + 2\mu), \qquad \bar{u}_{12} = \frac{u_{12}}{\phi_u},$$

$$\bar{u}_{21} = \frac{u_{21}}{\phi_u^2}, \qquad \bar{u}_{22} = \frac{1}{\phi_u^2}(u_{22} + 2\mu), \tag{74}$$

$$\bar{a}_1 = \frac{1}{V'}(a_1\phi_u - 2\phi_v), \qquad \bar{a}_2 = \frac{1}{(V')^2}(a_2\phi_u^2 - a_1\phi_u\phi_v + \phi_v^2),$$

where

$$\mu = \frac{\phi_{uuu}}{\phi_u} - \frac{3}{2}\left(\frac{\phi_{uu}}{\phi_u}\right)^2 = \{\phi, u\}. \tag{75}$$

We see that the net-invariants a_1 and a_2 are not invariants of the one-parameter family; but if we put, as before,

$$A = a_1^2 - 4a_2,$$

we find

$$\bar{A} = \frac{\phi_u^2 A}{(V')^2}, \tag{76}$$

so that A is such an invariant.

It is evident that $\theta_{4,0}$, $\theta_{9,0}$, $\theta_{10,0}$, and $\vartheta_{10,0}$ are invariants of the one-parameter family of ruled surfaces. The formulas (74) enable us to verify that the same thing is true of $\theta_{1,1}$. In order to investigate the effect of (72) on $\theta_{4,1}$, $\theta_{-4,6}$, $\theta_{1,2}$, and $\theta_{-2,4}$, we must first obtain formulas for $\bar{v}_{ik}$, $\bar{a}_{ik}^{(1)}$, and $\bar{a}_{ik}^{(2)}$.

If θ is any function of u and v, we have

$$\frac{\partial\theta}{\partial\bar{u}} = \frac{1}{\phi_u}\frac{\partial\theta}{\partial u}, \qquad \frac{\partial\theta}{\partial\bar{v}} = \frac{1}{V'}\left[-\frac{\phi_v}{\phi_u}\frac{\partial\theta}{\partial u} + \frac{\partial\theta}{\partial v}\right]. \tag{77}$$

Consequently we find

$$\bar{v}_{11} - \bar{v}_{22} = \frac{1}{\phi_u^3}\left[v_{11} - v_{22} - 4(u_{11} - u_{22})\frac{\phi_{uu}}{\phi_u}\right],$$

$$\bar{v}_{12} = \frac{1}{\phi_u^3}\left(v_{12} - 4u_{12}\frac{\phi_{uu}}{\phi_u}\right), \qquad \bar{v}_{21} = \frac{1}{\phi_u^3}\left(v_{21} - 4u_{21}\frac{\phi_{uu}}{\phi_u}\right), \tag{78}$$

by making use of (73), (74), (77), and the equations which define v_{ik}. From (73) and (77) we find

$$(\bar{a}_{11})_{\bar{u}} = \frac{1}{V'}\left[(a_{11})_u + \frac{\phi_{uu}}{\phi_u}a_{11} - \frac{\phi_{uv}}{\phi_u}\right],$$

$$(\bar{a}_{12})_{\bar{u}} = \frac{1}{V'}\left[(a_{12})_u + \frac{\phi_{uu}}{\phi_u}a_{12}\right],$$

$$(\bar{a}_{21})_{\bar{u}} = \frac{1}{V'}\left[(a_{21})_u + \frac{\phi_{uu}}{\phi_u}a_{21}\right],$$

$$(\bar{a}_{22})_{\bar{u}} = \frac{1}{V'}\left[(a_{22})_u + \frac{\phi_{uu}}{\phi_u}a_{22} - \frac{\phi_{uv}}{\phi_u}\right],$$

$$(V')^2(\bar{a}_{11})_{\bar{v}} = -\phi_v\left[(a_{11})_u + a_{11}\frac{\phi_{uu}}{\phi_u} - \frac{\phi_{uv}}{\phi_u}\right] + \phi_u\left[(a_{11})_v + a_{11}\frac{\phi_{uv}}{\phi_u} - \frac{\phi_{vv}}{\phi_u} - \zeta a_{11} + \zeta\frac{\phi_v}{\phi_u}\right],$$

(79)

$$(V')^2(\bar{a}_{12})_{\bar{v}} = -\phi_v\left[(a_{12})_u + a_{12}\frac{\phi_{uu}}{\phi_u}\right] + \phi_u\left[(a_{12})_v + a_{12}\frac{\phi_{uv}}{\phi_u} - \zeta a_{12}\right],$$

$$(V')^2(\bar{a}_{21})_{\bar{v}} = -\phi_v\left[(a_{21})_u + a_{21}\frac{\phi_{uu}}{\phi_u}\right] + \phi_u\left[(a_{21})_v + a_{21}\frac{\phi_{uv}}{\phi_u} - \zeta a_{21}\right],$$

$$(V')^2(\bar{a}_{22})_{\bar{v}} = -\phi_v\left[(a_{22})_u + a_{22}\frac{\phi_{uu}}{\phi_u} - \frac{\phi_{uv}}{\phi_u}\right] + \phi_u\left[(a_{22})_v + a_{22}\frac{\phi_{uv}}{\phi_u} - \frac{\phi_{vv}}{\phi_u} - \zeta a_{22} + \zeta\frac{\phi_v}{\phi_u}\right].$$

From the equations (73) and the definition of π_{ik}, we deduce the following:

$$V'\bar{\pi}_{11} = \pi_{11} - \frac{\phi_v}{\phi_u}p_{11} + \left(a_{11} - \frac{\phi_v}{\phi_u}\right)\frac{\phi_{uu}}{\phi_u},$$

(80) $$V'\bar{\pi}_{12} = \pi_{12} - \frac{\phi_v}{\phi_u}p_{12} + a_{12}\frac{\phi_{uu}}{\phi_u}, \qquad V'\bar{\pi}_{21} = \pi_{21} - \frac{\phi_v}{\phi_u}p_{21} + a_{21}\frac{\phi_{uu}}{\phi_u},$$

$$V'\bar{\pi}_{22} = \pi_{22} - \frac{\phi_v}{\phi_u}p_{22} + \left(a_{22} - \frac{\phi_v}{\phi_u}\right)\frac{\phi_{uu}}{\phi_u}.$$

Making use of (79) and (80), we now find

$$V'\bar{a}_{11}^{(1)} = a_{11}^{(1)} + 2a_{11}\frac{\phi_{uu}}{\phi_u} - 2\frac{\phi_{uv}}{\phi_u}, \qquad V'\bar{a}_{12}^{(1)} = a_{12}^{(1)} + 2a_{12}\frac{\phi_{uu}}{\phi_u},$$

(81)

$$V'\bar{a}_{21}^{(1)} = a_{21}^{(1)} + 2a_{21}\frac{\phi_{uu}}{\phi_u}, \qquad V'\bar{a}_{22}^{(1)} = a_{22}^{(1)} + 2a_{22}\frac{\phi_{uu}}{\phi_u} - 2\frac{\phi_{uv}}{\phi_u},$$

and

$$(V')^2\,\bar{a}_{11}^{(2)} = \phi_u\,a_{11}^{(2)} - \phi_v{}^2 a_{11}^{(1)} + 2\left(\phi_{uv} - \phi_v\frac{\phi_{uu}}{\phi_u} - \phi_u\zeta\right)a_{11} + 2\phi_v\left(\frac{\phi_{uv}}{\phi_u} - \frac{\phi_{vv}}{\phi_v} + \zeta\right),$$

$$(V')^2\,\bar{a}_{12}^{(2)} = \phi_u\,a_{12}^{(2)} - \phi_v\,a_{12}^{(1)} + 2\left(\phi_{uv} - \phi_v\frac{\phi_{uu}}{\phi_u} - \phi_u\zeta\right)a_{12},$$

(82)

$$(V')^2\,\bar{a}_{21}^{(2)} = \phi_u\,a_{21}^{(2)} - \phi_v\,a_{21}^{(1)} + 2\left(\phi_{uv} - \phi_v\frac{\phi_{uu}}{\phi_u} - \phi_u\zeta\right)a_{21},$$

$$(V')^2\,\bar{a}_{22}^{(2)} = \phi_u\,a_{22}^{(2)} - \phi_v\,a_{22}^{(1)} + 2\left(\phi_{uv} - \phi_v\frac{\phi_{uu}}{\phi_u} - \phi_u\zeta\right)a_{22} + 2\phi_v\left(\frac{\phi_{uv}}{\phi_u} - \frac{\phi_{vv}}{\phi_v} + \zeta\right).$$

These formulas enable us to verify, at a glance, that the net-invariants $\theta_{4,\,1}$ and $\theta_{-2,\,4}$ are invariants of the one-parameter family as well. It remains to determine the effect of the transformation on the net-invariants $\theta_{-4,\,6}$ and $\theta_{1,\,2}$, and to find two further invariants of the one-parameter family from these two net-invariants.

This may be accomplished as follows. Equations (74) show that we can make $\bar{a}_1 = 0$ by choosing for our new variable $\bar{u} = \phi(u, v)$ a solution (different from $\phi = \text{const.}$) of the differential equation

$$a_1\,\phi_u - 2\phi_v = 0. \tag{83}$$

Moreover, the most general solution of (83) will be an arbitrary function of $\bar{u}$. Consequently, the one-parameter family of ruled surfaces $v = \text{const.}$ determines uniquely a second one-parameter family of ruled surfaces $\bar{u} = \text{const.}$, the two families constituting a net for which the net-invariant a_1 has the value zero. We shall call this second one-parameter family of ruled surfaces the *conjugate* family. Since it is determined uniquely by the original family, it is clear that the invariants of the *net* formed by the given one-parameter family and its conjugate family will be invariants of the one-parameter family as well. We shall call such a net *a conjugate net of ruled surfaces*, and we shall explain later the geometric relation between the ruled surfaces of the two conjugate families.

From every net-invariant we obtain an invariant of the one-parameter family, by subjecting it to the general transformation of the form (72), and then substituting in the expression thus found the following values, derived from (83);

$$\phi_v = \tfrac{1}{2}\, a_1\, \phi_u, \qquad \phi_{uv} = \tfrac{1}{2}\,[\,(a_1)_u\, \phi_u + a_1\, \phi_{uu}\,],$$
$$\phi_{vv} = \tfrac{1}{2}\,\{(a_1)_v + \tfrac{1}{2}\, a_1\,(a_1)_u\}\phi_u + \tfrac{1}{4}\, a_1^2\, \phi_{uu}, \tag{84}$$

whence

$$\phi_{uv} - \phi_v \frac{\phi_{uu}}{\phi_u} - \phi_u\, \zeta = \tfrac{1}{2}\, \phi_u\,[\,(a_1)_u - 2\zeta\,],$$
$$\frac{\phi_{uv}}{\phi_u} - \frac{\phi_{vv}}{\phi_v} + \zeta = \zeta - \frac{(a_1)_v}{a_1}. \tag{85}$$

The resulting expression will be an invariant of the one-parameter family, except for a factor of the form $\phi_u^k (V')^l$.

Let us apply this process to the net-invariant $\theta_{-4,\,6}$. We have

$$\theta_{-4,\,6} = AA^{(2)} - (A_v)^2,$$

where

$$A^{(2)} = (a_1^{(2)})^2 - 4a_2^{(2)} = (a_{11}^{(2)} - a_{22}^{(2)})^2 + 4a_{12}^{(2)}\, a_{21}^{(2)}.$$

According to (82) and (85) we have

$$\frac{(V')^2}{\phi_u}(\bar{a}_{11}^{(2)} - \bar{a}_{22}^{(2)}) = a_{11}^{(2)} - a_{22}^{(2)} - \tfrac{1}{2}\, a_1\,(a_{11}^{(1)} - a_{22}^{(1)}) + [\,(a_1)_u - 2\zeta\,](a_{11} - a_{22}),$$

etc.,

so that

$$\begin{aligned}\frac{(V')^4}{\phi_u^2}\,\bar{A}^{(2)} = {} & [\,a_{11}^{(2)} - a_{22}^{(2)} - \tfrac{1}{2}\, a_1\,(a_{11}^{(1)} - a_{22}^{(1)}) + (a_1)_u\,(a_{11} - a_2)\,]^2 \\ & + 4\,[\,a_{12}^{(2)} - \tfrac{1}{2}\, a_1\, a_{12}^{(1)} + (a_1)_u\, a_{12}\,]\,[\,a_{21}^{(2)} - \tfrac{1}{2}\, a_1\, a_{21}^{(1)} + (a_1)_u\, a_{21}\,] \\ & - 4\zeta\,[\,A_v - \tfrac{1}{2}\, a_1\, A_u + (a_1)_u\, A\,] + 4\zeta^2 A,\end{aligned} \tag{86}$$

since we have

$$\begin{aligned}(a_{11} - a_{22})\,(a_{11}^{(1)} - a_{22}^{(1)}) + 2a_{12}\, a_{21}^{(1)} + 2a_{21}\, a_{12}^{(1)} &= A_u, \\ (a_{11} - a_{22})\,(a_{11}^{(2)} - a_{22}^{(2)}) + 2a_{12}\, a_{21}^{(2)} + 2a_{21}\, a_{12}^{(2)} &= A_v.\end{aligned} \tag{87}$$

We also have

$$\bar{A} = \frac{\phi_u^3}{(V')^2}\, A,$$

whence

$$\frac{(V')^3}{\phi_u^2}\,\bar{A}_{\bar{v}} = -\frac{\phi_v}{\phi_u} A_u + A_v + 2\left(\frac{\phi_{uv}}{\phi_u} - \frac{\phi_v}{\phi_u}\frac{\phi_{uu}}{\phi_u} - \zeta\right) A, \tag{88}$$

or, under the conditions (84) and (85),

$$\frac{(V')^3}{\phi_u^2}\,\bar{A}_{\bar{v}} = A_v - \tfrac{1}{2}\, a_1\, A_u + (a_1)_u\, A - 2\zeta A. \tag{89}$$

Consequently we find for

$$\frac{(V')^6}{\phi_u^2}\,\bar{\theta}_{-4,\,6}$$

the following expression which is the new invariant which we were seeking

$$\theta'_{-4,6} = A\,[\{a_{11}^{(2)} - a_{22}^{(2)} - \tfrac{1}{2}\,a_1(a_{11}^{(1)} - a_{22}^{(1)}) + (a_1)_u(a_{11} - a_{22})\}^2 + 4\{a_{12}^{(2)} - \tfrac{1}{2}\,a_1\,a_{12}^{(1)} + (a_1)_u\,a_{12}\}\{a_{21}^{(2)} - \tfrac{1}{2}\,a_1\,a_{21}^{(1)} + (a_1)_u\,a_{21}\}] - [A_v - \tfrac{1}{2}\,a_1\,A_u + (a_1)_u\,A]^2,$$

and which, on account of (87) may be written more simply

$$\theta'_{-4,6} = A\,[\{a_{11}^{(2)} - a_{22}^{(2)} - \tfrac{1}{2}\,a_1(a_{11}^{(1)} - a_{22}^{(1)})\}^2 + 4\{a_{12}^{(2)} - \tfrac{1}{2}\,a_1\,a_{12}^{(1)}\}\{a_{21}^{(2)} - \tfrac{1}{2}\,a_1\,a_{21}^{(1)}\}] - (A_v - \tfrac{1}{2}\,a_1\,A_u)^2. \tag{90}$$

The formula for the net-invariant $\theta_{1,2}$ was

$$\theta_{1,2} = (a^{(2)}, u) - 2(a, u)_v.$$

Making use of (84) and (85), we find

$$I''\,\phi_u(\overline{a, u}) = (a, u) - \tfrac{1}{2}\,a_1\,I = \theta_{1,1},$$

whence, according to (77),

$$\phi_u(I'')^2(\overline{a, u})_{\bar v} = (\theta_{1,1})_v - \tfrac{1}{2}\,a_1(\theta_{1,1})_u - \tfrac{1}{2}(a_1)_u\,\theta_{1,1} - \zeta\theta_{1,1}. \tag{91}$$

On the other hand, we find

$$\phi_u(I'')^2(\overline{a^{(2)}, u}) = (a^{(2)}, u) - \tfrac{1}{2}\,a_1(a^{(1)}, u) + (a_1)_u(a, u) - (a_1)_v\,I - 2\zeta\theta_{1,1}, \tag{92}$$

giving rise to the new invariant

$$\theta'_{1,2} = (a^{(2)}, u) - \tfrac{1}{2}\,a_1(a^{(1)}, u) - I\,[(a_1)_v - \tfrac{1}{2}\,a_1(a_1)_u] - 2\,[(\theta_{1,1})_v - \tfrac{1}{2}\,a_1(\theta_{1,1})_u] + 2(a_1)_u\,\theta_{1,1}. \tag{93}$$

The ten invariants

$$\theta_{0,4},\ \theta_{0,9},\ \theta_{0,10},\ \vartheta_{0,10},\ A,\ \theta_{1,1},\ \theta_{4,1},\ \theta'_{-4,6},\ \theta'_{1,2},\ \theta_{-2,4} \tag{94}$$

clearly determine a one-parameter family of ruled surfaces, except for projective transformations. For, if we specify besides the value of a_1 as a function of u and v, for instance $a_1 = 0$, the corresponding net of ruled surfaces is determined except for a projective transformation.

9. Introduction of the focal surfaces as surfaces of reference. The focal canonical form

The generators of the ruled surfaces of a one-parameter family form a congruence of lines. The invariants of the congruence are those invariants of the one-parameter family which are left unaltered by any transformation of the form

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v), \tag{95}$$

where both ϕ and ψ are arbitrary functions of u and v. These invariants might be obtained by extending the methods of Article 8 to the more general transformation (95). It seems preferable however to establish connection, at this point, with a previously developed projective theory of congruences.* We accomplish this by showing how to transform system (S) into the system of differential equations which was used in that other theory.

Let us assume that the quadratic equation

$$-a_{21}t^2 + (a_{11} - a_{22})t + a_{12} = 0, \tag{96}$$

has two distinct roots, so that

$$A = (a_{11} - a_{22})^2 + 4a_{12}a_{21}$$

is different from zero. If we make a transformation of form (22), in which the ratios $\alpha : \gamma$ and $\beta : \delta$ are equated to the two roots of (96), we shall have $\alpha\delta - \beta\gamma \neq 0$, and according to (26), the transformed system will have $\bar{a}_{12} = \bar{a}_{21} = 0$.

Although this transformation is not unique, the resulting new surfaces of reference, $S_{\bar{y}}$ and $S_{\bar{z}}$, are determined uniquely; they are, in fact, the focal surfaces of the congruence. This follows from the fact that the first order equations of the transformed system (S) will have the form

$$\bar{y}_v = \bar{a}_{11}\bar{y}_u + \bar{b}_{11}\bar{y} + \bar{b}_{12}\bar{z}, \qquad \bar{z}_v = \bar{a}_{22}\bar{z}_u + \bar{b}_{21}\bar{y} + \bar{b}_{22}\bar{z}.$$

For, if $\bar{b}_{12}$ and $\bar{b}_{21}$ are not zero, these equations show that every line $P_{\bar{y}}P_{\bar{z}}$ of the congruence is tangent to each of the surfaces $S_{\bar{y}}$ and $S_{\bar{z}}$. The cases $\bar{b}_{12} = 0$ or $\bar{b}_{21} = 0$ correspond to the cases when one or both focal surfaces degenerate into curves. Consequently $\bar{b}_{12}$ and $\bar{b}_{21}$ must be invariants of the congruence whose explicit expressions we shall obtain very soon.

We may formulate our result more elegantly as follows. *The function*

$$a_{21}y^2 - (a_{11} - a_{22})yz - a_{12}z^2 \tag{97}$$

is a quadratic covariant of system (S). *The factors of this covariant determine in general two points on each line of the congruence, the foci of the line, that is, the points where the line touches the focal locus of the congruence. The foci are distinct or coincident according as* A *is not or is equal to zero.*

From this last remark it is clear that A is an invariant of the congruence.

Let us factor the covariant (97), and denote its factors by $\bar{y}$ and $\bar{z}$. Then $S_{\bar{y}}$ and $S_{\bar{z}}$ will be the two sheets of the focal locus. In order to avoid exceptional cases, and to preserve symmetry, we write the factors of (97) in the following form

* Brussels Paper.

$$\bar{y} = \sqrt{a}y + \frac{a_{12}}{\sqrt{a}}z, \qquad \bar{z} = \frac{-a_{21}}{\sqrt{a}}y + \sqrt{a}z, \tag{98}$$

where

$$a = \tfrac{1}{2}(a_{11} - a_{22} + \sqrt{A}). \tag{99}$$

In these formulas $\sqrt{A}$ stands for that one of the two square roots of

$$A = (a_{11} - a_{22})^2 + 4a_{12}a_{21}$$

which reduces to $a_{11} - a_{22}$ when a_{12} or a_{21} is equal to zero. The square root $\sqrt{a}$ may be chosen in either of the two available ways; but after a choice has been made, it must be adhered to. Since we are discussing the case $A \neq 0$, a will also be different from zero as a result of the specification just made of the meaning of the symbol $\sqrt{A}$.

From (98) we find

$$\sqrt{A}y = \sqrt{a}\bar{y} - \frac{a_{12}}{\sqrt{a}}\bar{z}, \qquad \sqrt{A}z = \frac{a_{21}}{\sqrt{a}}\bar{y} + \sqrt{a}\bar{z}. \tag{100}$$

Consequently we shall find the coefficients A_{ik}, B_{ik}, P_{ik}, Q_{ik} of the resulting canonical form of the system (S) for $\bar{y}$ and $\bar{z}$, by substituting into the general formulas of Article 5 the values

$$\alpha = \sqrt{\frac{a}{A}} = \frac{a}{\sqrt{Aa}}, \qquad \beta = \frac{-a_{12}}{\sqrt{Aa}}, \qquad \gamma = \frac{a_{21}}{\sqrt{Aa}}, \qquad \delta = \frac{a}{\sqrt{Aa}}.$$

This canonical form shall be called the *focal* canonical form.

We find in this way

$$\begin{aligned}
\sqrt{A}U_{11} &= \tfrac{1}{2}(a_{11} - a_{22})(u_{11} - u_{22}) \\
&\qquad + a_{21}u_{12} + a_{12}u_{21} + \tfrac{1}{2}\sqrt{A}(u_{11} + u_{22}), \\
\sqrt{A}U_{22} &= -\tfrac{1}{2}(a_{11} - a_{22})(u_{11} - u_{22}) \\
&\qquad - a_{21}u_{12} - a_{12}u_{21} + \tfrac{1}{2}\sqrt{A}(u_{11} + u_{22}), \\
\sqrt{A}U_{12} &= -a_{12}(u_{11} - u_{22}) + au_{12} - \frac{a_{12}^2}{a}u_{21}, \\
\sqrt{A}U_{21} &= -a_{21}(u_{11} - u_{22}) - \frac{a_{21}^2}{a}u_{12} + au_{21},
\end{aligned} \tag{101}$$

and

$$\begin{aligned}
A_{11} &= \tfrac{1}{2}(a_{11} + a_{22} + \sqrt{A}), \qquad A_{22} = \tfrac{1}{2}(a_{11} + a_{22} - \sqrt{A}), \\
A_{12} &= A_{21} = 0.
\end{aligned} \tag{102}$$

The transformations of form (22) transform the quantities a_{ik} and u_{ik} cogrediently. Comparison of (24) and (26) shows that the p_{ik}'s are transformed cogrediently with the u_{ik}'s except for the presence in each $\bar{p}_{ik}$ of some

terms which involve partial derivatives of $\alpha, \beta, \gamma, \delta$ with respect to u. Therefore $\sqrt{A}P_{ik}$ will be obtained from $\sqrt{A}U_{ik}$ if we replace the u_{ik}'s which appear in the right members of (101) by the corresponding p_{ik}'s and then add the additional terms involving the derivatives of $\alpha, \beta, \gamma, \delta$.

We find the following formulas;

$$\begin{aligned}
\sqrt{A}P_{11} &= \frac{a_u}{a^2}(a^2 - a_{12}a_{21}) - \frac{A_u}{aA}(a^2 + a_{12}a_{21}) + \frac{2a_{12}(a_{21})_u}{a} \\
&\quad + \tfrac{1}{2}(a_{11} - a_{22})(p_{11} - p_{22}) \\
&\quad + a_{21}p_{12} + a_{12}p_{21} + \tfrac{1}{2}\sqrt{A}(p_{11} + p_{22}), \\
\sqrt{A}P_{22} &= \frac{a_u}{a^2}(a^2 - a_{12}a_{21}) - \frac{A_u}{aA}(a^2 + a_{12}a_{21}) + \frac{2a_{21}(a_{12})_u}{a} \\
&\quad - \tfrac{1}{2}(a_{11} - a_{22})(p_{11} - p_{22}) \\
&\quad - a_{21}p_{12} - a_{12}p_{21} + \tfrac{1}{2}\sqrt{A}(p_{11} + p_{22}), \\
\sqrt{A}P_{12} &= \frac{2}{a}[a_{12}a_u - a(a_{12})_u] - a_{12}(p_{11} - p_{22}) + ap_{12} - \frac{a_{12}^2}{a}p_{21}, \\
\sqrt{A}P_{21} &= -\frac{2}{a}[a_{21}a_u - a(a_{21})_u] - a_{21}(p_{11} - p_{22}) - \frac{a_{21}^2}{a}p_{12} + ap_{21}.
\end{aligned} \tag{103}$$

We notice the relations

$$a^2 - a_{12}a_{21} = a(a_{11} - a_{22}), \qquad a^2 + a_{12}a_{21} = \tfrac{1}{2}a\sqrt{A}, \tag{104}$$

and find

$$P_{11} + P_{22} = p_{11} + p_{22} - \frac{A_u}{A}, \tag{105}$$

a result also deducible from (24) and the value of

$$\Delta = \alpha\delta - \beta\gamma = \frac{1}{\sqrt{A}}. \tag{106}$$

We find further

$$\begin{aligned}
\sqrt{A}(P_{11} - P_{22}) &= \frac{2}{a}[a_{12}(a_{21})_u - a_{21}(a_{12})_u] \\
&\quad + (a_{11} - a_{22})(p_{11} - p_{22}) + 2(a_{21}p_{12} + a_{12}p_{21}).
\end{aligned} \tag{107}$$

These formulas assume a more compact form if we introduce the quantities $a_{ik}^{(1)}$ instead of the partial derivatives $(a_{ik})_u$. We find

$$
\begin{aligned}
P_{11} &= p_{11} - \frac{A_u}{2A} + \frac{p_{12}\,a_{21} + p_{21}\,a_{12}}{2a} + \frac{a_{12}\,a_{21}^{(1)} - a_{21}\,a_{12}^{(1)}}{2a\sqrt{A}},\\
P_{22} &= p_{22} - \frac{A_u}{2A} - \frac{p_{12}\,a_{21} + p_{21}\,a_{12}}{2a} - \frac{a_{12}\,a_{21}^{(1)} - a_{21}\,a_{12}^{(1)}}{2a\sqrt{A}},\\
P_{12} &= -\frac{a_{12}^{(1)}}{\sqrt{A}} + \frac{a_{12}}{2a}\left[\frac{A_u}{A} + \frac{a_{11}^{(1)} - a_{22}^{(1)}}{\sqrt{A}}\right],\\
P_{21} &= +\frac{a_{21}^{(1)}}{\sqrt{A}} - \frac{a_{21}}{2a}\left[\frac{A_u}{A} + \frac{a_{11}^{(1)} - a_{22}^{(1)}}{\sqrt{A}}\right].
\end{aligned}
\tag{108}
$$

On account of the frequently noted cogredience relations between the p_{ik}'s and the π_{ik}'s, we have similarly

$$
\begin{aligned}
\Pi_{11} &= \pi_{11} - \frac{A_v}{2A} + \frac{\pi_{12}\,a_{21} + \pi_{21}\,a_{12}}{2a} + \frac{a_{12}\,a_{21}^{(2)} - a_{21}\,a_{12}^{(2)}}{2a\sqrt{A}},\\
\Pi_{22} &= \pi_{22} - \frac{A_v}{2A} - \frac{\pi_{12}\,a_{21} + \pi_{21}\,a_{12}}{2a} - \frac{a_{12}\,a_{21}^{(2)} - a_{21}\,a_{12}^{(2)}}{2a\sqrt{A}},\\
\Pi_{12} &= -\frac{a_{12}^{(2)}}{\sqrt{A}} + \frac{a_{12}}{2a}\left[\frac{A_v}{A} + \frac{a_{11}^{(2)} - a_{22}^{(2)}}{\sqrt{A}}\right],\\
\Pi_{21} &= +\frac{a_{21}^{(2)}}{\sqrt{A}} - \frac{a_{21}}{2a}\left[\frac{A_v}{A} + \frac{a_{11}^{(2)} - a_{22}^{(2)}}{\sqrt{A}}\right].
\end{aligned}
\tag{109}
$$

The remaining coefficients, B_{ik} and Q_{ik}, of the focal canonical form are expressible in terms of those already computed by means of the formulas

$$
\begin{aligned}
2B_{11} &= A_{11}\,P_{11} - \Pi_{11}, \qquad 2B_{12} = A_{11}\,P_{12} - \Pi_{12},\\
2B_{21} &= A_{22}\,P_{21} - \Pi_{21}, \qquad 2B_{22} = A_{22}\,P_{22} - \Pi_{22},
\end{aligned}
\tag{110}
$$

and

$$
\begin{aligned}
4Q_{11} &= 2(P_{11})_u + P_{11}^2 + P_{12}\,P_{21} - U_{11},\\
4Q_{12} &= 2(P_{12})_u + P_{12}(P_{11} + P_{22}) - U_{12},\\
4Q_{21} &= 2(P_{21})_u + P_{21}(P_{11} + P_{22}) - U_{21},\\
4Q_{22} &= 2(P_{22})_u + P_{22}^2 + P_{12}\,P_{21} - U_{22}.
\end{aligned}
\tag{111}
$$

10. Determination of the developables of the congruence

The first order equations of system (S) were given originally in the form

$$
\begin{aligned}
y_v &= a_{11}\,y_u + a_{12}\,z_u + b_{11}\,y + b_{12}\,z,\\
z_v &= a_{21}\,y_u + a_{22}\,z_u + b_{21}\,y + b_{22}\,z.
\end{aligned}
\tag{112}
$$

Let us transform these equations by putting

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v), \tag{113}$$

where ϕ and ψ are arbitrary functions of u and v. The geometrical effect of this transformation is to replace the net of ruled surfaces formed by $u = \text{const.}$ and $v = \text{const.}$, by any other net $\bar{u} = \text{const.}$ and $\bar{v} = \text{const.}$, formed from the lines of the same congruence.

As a result of (113) we find

$$y_u = \frac{\partial y}{\partial \bar{u}}\phi_u + \frac{\partial y}{\partial \bar{v}}\psi_u, \qquad y_v = \frac{\partial y}{\partial \bar{u}}\phi_v + \frac{\partial y}{\partial \bar{v}}\psi_v, \text{ etc.}, \tag{114}$$

where we assume

$$\phi_u \psi_v - \phi_v \psi_u \neq 0. \tag{115}$$

Consequently (112) becomes

$$\begin{aligned} \frac{\partial y}{\partial \bar{v}} &= \bar{a}_{11}\frac{\partial y}{\partial \bar{u}} + \bar{a}_{12}\frac{\partial z}{\partial \bar{u}} + \bar{b}_{11}\, y + \bar{b}_{12}\, z, \\ \frac{\partial z}{\partial \bar{v}} &= \bar{a}_{21}\frac{\partial y}{\partial \bar{u}} + \bar{a}_{22}\frac{\partial z}{\partial \bar{u}} + \bar{b}_{21}\, y + \bar{b}_{22}\, z, \end{aligned} \tag{116}$$

where

$$\begin{aligned} \Delta\bar{a}_{11} &= -(a_{11}a_{22} - a_{12}a_{21})\phi_u\psi_u + a_{11}\phi_u\psi_v + a_{22}\phi_v\psi_u - \phi_v\psi_v, \\ \Delta\bar{a}_{22} &= -(a_{11}a_{22} - a_{12}a_{21})\phi_u\psi_u + a_{22}\phi_u\psi_v + a_{11}\phi_v\psi_u - \phi_v\psi_v, \\ \Delta\bar{a}_{12} &= a_{12}(\phi_u\psi_v - \phi_v\psi_u), \qquad \Delta\bar{a}_{21} = a_{21}(\phi_u\psi_v - \phi_v\psi_u), \\ \Delta\bar{b}_{11} &= (a_{12}b_{21} - a_{22}b_{11})\psi_u + b_{11}\psi_v, \\ \Delta\bar{b}_{12} &= (a_{12}b_{22} - a_{22}b_{12})\psi_u + b_{12}\psi_v, \\ \Delta\bar{b}_{21} &= (a_{21}b_{11} - a_{11}b_{21})\psi_u + b_{21}\psi_v, \\ \Delta\bar{b}_{22} &= (a_{21}b_{12} - a_{11}b_{22})\psi_u + b_{22}\psi_v, \\ \Delta &= (a_{11}a_{22} - a_{12}a_{21})\psi_u^2 - (a_{11} + a_{22})\psi_u\psi_v + \psi_v^2. \end{aligned} \tag{117}$$

As long as Δ is different from zero, the equations (116) are of the same form as (112). If $\Delta = 0$, the two equations (116) reduce to an equation of the form

$$\alpha\frac{\partial y}{\partial \bar{u}} + \beta\frac{\partial z}{\partial \bar{u}} + \gamma y + \delta z = 0,$$

showing that the ruled surfaces $\bar{v} = \text{const.}$ are developables.*

If $A = (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})$ is different from zero, the condition $\Delta = 0$ can be satisfied in two essentially distinct ways, by integrating the differential equations for ψ which are obtained by equating to zero the

* *Proj. Diff. Geom.*, p. 131.

factors of Δ. This proves the familiar theorem that a congruence with distinct focal surfaces has two distinct one-parameter families of developables.

If $A = a_1^2 - 4a_2$ is equal to zero, the two sheets of the focal surface coincide, and there exists only a single one-parameter family of developables in the congruence. Let us assume that the surface of reference S_y is so chosen as to coincide with the focal surface. Then we shall have

$$a_{12} = 0, \qquad a_{11} - a_{22} = 0, \qquad a_{21} \neq 0, \tag{118}$$

since the covariant

$$a_{21}\, y^2 - (a_{11} - a_{22})\, yz - a_{12}\, z^2$$

must reduce to a multiple of y^2. Then Δ is a perfect square, and the differential equation $\Delta = 0$ reduces to

$$a_{11}\, \psi_u - \psi_v = 0. \tag{119}$$

If $\psi(u, v)$ is a solution of this equation, the ruled surfaces $\psi(u, v) = \text{const.}$ will be the developables of the congruence.

On the other hand, if we assume $b_{12} \neq 0$, equations (17) show that the differential equation of the asymptotic lines on S_y is

$$-p_{12}\, du^2 + 2c_{12}\, du dv + e_{12}\, dv^2 = 0, \tag{120}$$

and, in our case, we have

$$c_{12} = -a_{11}\, p_{12} + b_{12}, \qquad e_{12} = a_{11}\, c_{12} + b_{12}\, a_{11} = -a_{11}^2\, p_{12} + 2a_{11}\, b_{12}.$$

Consequently (120) becomes

$$(du + a_{11}\, dv)\,[-p_{12}\,(du + a_{11}\, dv) + 2b_{12}\, dv] = 0. \tag{121}$$

But an integral of

$$du + a_{11}\, dv = 0$$

is a solution of (119), giving rise to the familiar theorem that a congruence with coincident focal surfaces is composed of the tangents of one of the two families of asymptotic lines on its focal surface. The assumption $b_{12} \neq 0$ upon which we have based the proof, merely requires that the focal surface do not degenerate into a curve. Equation (121) makes it convenient to examine the second set of asymptotic lines of the focal surface and the corresponding congruence of tangents.

If the two focal sheets coincide and at the same time degenerate into coincident curves, we may assume

$$a_{11} = 0, \qquad a_{12} = 0, \qquad a_{22} = 0, \qquad a_{21} \neq 0, \qquad b_{12} = 0$$

if we use the coincident focal curves for the locus of P_y, and the developables of the congruence as surfaces $u = \text{const.}$ We shall then have

$$y_v = b_{11}\, y\,, \qquad z_v = b_{21}\, y + b_{22}\, z\,.$$

These equations show that P_y remains fixed when v alone changes, and that P_z can only move along the straight line joining it to P_y when v alone changes. If v remains constant, and u only changes, P_y moves on the double focal curve F, and the plane whose homogeneous coördinates are given by the four third order determinants of the form

$$\lambda = |y, z, y_u|$$

contains P_z and the tangent of F at P_y. We find

$$\lambda_v = (2b_{11} + b_{22})\lambda\,.$$

Consequently this plane does not change when v changes; it changes only with u. Consequently with the variation of both variables, this plane generates a developable which contains the curve F. The congruence may therefore be described as follows. On an arbitrary developable draw any curve. The lines tangent to the developable along this curve constitute a congruence of this sort. Clearly the developables of this congruence are the ∞^1 pencils whose vertices are the points of the given curve, and whose planes are the tangent planes of the fundamental developable.

There remains one case which requires special attention. If

$$a_{12} = a_{21} = a_{11} - a_{22} = 0\,,$$

the foci are entirely indeterminate, but the developables are not, and we may assume

$$a_{11} = a_{22} = a_{12} = a_{21} = 0\,.$$

The equations

$$y_v = b_{11}\, y + b_{12}\, z\,, \qquad z_v = b_{21}\, y + b_{22}\, z$$

show that P_y and P_z move along the same fixed straight line if v varies while u remains constant. Thus the congruence, in this case degenerates into a one-parameter family of straight lines. All of the ruled surfaces of the family $v =$ const. coincide.

11. The differential equations of the congruence referred to its focal sheets and developables

We cannot introduce the developables of the congruence as parametric ruled surfaces and also preserve the form (S) of our system of differential equations. We now proceed to show how (S) will be transformed.

Let us start with the first order equations of (S) written in the focal canonical form

$$\begin{aligned} \bar{y}_v &= A_{11}\,\bar{y}_u + B_{11}\,\bar{y} + B_{12}\,\bar{z}\,, \\ \bar{z}_v &= A_{22}\,\bar{z}_u + B_{21}\,\bar{y} + B_{22}\,\bar{z}\,, \end{aligned} \qquad \sqrt{A} = A_{11} - A_{22} \neq 0\,, \tag{122}$$

so that $S_{\bar{y}}$ and $S_{\bar{z}}$ are the focal sheets, assumed to be distinct, but not necessarily non-degenerate. The expressions for A_{ik} and B_{ik} in terms of the coefficients of (S) are given in Article 9.

Since $A_{12} = A_{21} = 0$, the developables of the congruence are obtained by equating to constants the functions

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v), \tag{123}$$

where ϕ and ψ satisfy the conditions

$$\phi_v = A_{11}\phi_u, \qquad \psi_v = A_{22}\psi_u, \tag{124}$$

respectively, and where constant solutions of these equations are, of course, excluded. We may, therefore, assume $\phi_u \neq 0$, $\psi_u \neq 0$.

If we introduce $\bar{u}$ and $\bar{v}$, as defined by (123) and (124), as independent variables, the equations (122) assume the form

$$\frac{\partial \bar{y}}{\partial \bar{v}} = m'\bar{y} + m''\bar{z}, \qquad \frac{\partial \bar{z}}{\partial \bar{u}} = n'\bar{y} + n''\bar{z}, \tag{125}$$

where

$$m' = -\frac{\beta_{11}}{\psi_u}, \qquad m'' = -\frac{\beta_{12}}{\psi_u}, \qquad n' = \frac{\beta_{21}}{\phi_u}, \qquad n'' = \frac{\beta_{22}}{\phi_u}, \qquad \beta_{ik} = \frac{B_{ik}}{\sqrt{A}}. \tag{126}$$

A transformation of the form

$$\bar{y} = \omega\eta, \qquad \bar{z} = \omega'\zeta \tag{127}$$

will not change the surfaces of reference. Its effect upon the fundamental quantities entering into the focal canonical form is given by the equations

$$\begin{aligned}
&\bar{P}_{11} = P_{11} + 2\frac{\omega_u}{\omega}, && \bar{P}_{12} = \frac{\omega'}{\omega}P_{12}, \\
&P_{21} = \frac{\omega}{\omega'}P_{21}, && \bar{P}_{22} = P_{22} + 2\frac{\omega'_u}{\omega'}, \\
&\bar{U}_{11} = U_{11}, \quad \bar{U}_{12} = \frac{\omega'}{\omega}U_{12}, && \bar{U}_{21} = \frac{\omega}{\omega'}U_{21}, \quad \bar{U}_{22} = U_{22}, \\
&\bar{A}_{11} = A_{11}, \quad \bar{A}_{12} = 0, && \bar{A}_{21} = 0, \quad \bar{A}_{22} = A_{22}, \\
&\bar{\Pi}_{11} = \Pi_{11} + 2\frac{\omega_v}{\omega}, && \bar{\Pi}_{12} = \frac{\omega'}{\omega}\Pi_{12}, \\
&\bar{\Pi}_{21} = \frac{\omega}{\omega'}\Pi_{21}, && \bar{\Pi}_{22} = \Pi_{22} + 2\frac{\omega'_v}{\omega'}, \\
&\bar{B}_{11} = B_{11} + A_{11}\frac{\omega_u}{\omega} - \frac{\omega_v}{\omega}, && \bar{B}_{12} = \frac{\omega'}{\omega}B_{12}, \\
&\bar{B}_{21} = \frac{\omega}{\omega'}B_{21}, && \bar{B}_{22} = B_{22} + A_{22}\frac{\omega'_u}{\omega'} - \frac{\omega'_v}{\omega'}.
\end{aligned} \tag{128}$$

The equations for $\bar{B}_{11}$ and $\bar{B}_{22}$ show that these quantities may be equated to zero provided that ω and ω' are chosen as solutions of the equations

$$A_{11}\frac{\omega_u}{\omega}-\frac{\omega_v}{\omega}+B_{11}=0, \qquad A_{22}\frac{\omega'_u}{\omega'}-\frac{\omega'_v}{\omega'}+B_{22}=0 \tag{129}$$

respectively. If we apply the transformation (127) directly to equations (126) we find the conditions

$$\frac{\partial \log \omega}{\partial \bar{v}}=-\frac{B_{11}}{\psi_u\sqrt{A}}=m', \qquad \frac{\partial \log \omega'}{\partial \bar{u}}=\frac{+B_{22}}{\phi_u\sqrt{A}}=n'' \tag{130}$$

equivalent to (129). Consequently this simplification of (125), resulting in equations of the form

$$\frac{\partial \eta}{\partial \bar{v}}=m\zeta, \qquad \frac{\partial \zeta}{\partial \bar{u}}=n\eta, \tag{131}$$

where

$$m=-\frac{\omega'}{\omega}\frac{B_{12}}{\psi_u\sqrt{A}}=\frac{\omega'}{\omega}m'', \qquad n=\frac{\omega}{\omega'}\frac{B_{21}}{\phi_u\sqrt{A}}=\frac{\omega}{\omega'}n', \tag{132}$$

may be accomplished by quadratures after the differential equations (124) which determine the developables of the congruence have been integrated.

The second order differential equations of (S) in its focal canonical form are

$$\begin{aligned}&\bar{y}_{uu}+P_{11}\bar{y}_u+P_{12}\bar{z}_u+Q_{11}\bar{y}+Q_{12}\bar{z}=0,\\ &\bar{z}_{uu}+P_{21}\bar{y}_u+P_{22}\bar{z}_u+Q_{21}\bar{y}+Q_{22}\bar{z}=0.\end{aligned} \tag{133}$$

Let us introduce the variables $\bar{u}$ and $\bar{v}$ into these equations. We have

$$\begin{aligned}\bar{y}_u&=\bar{y}_{\bar{u}}\phi_u+\bar{y}_{\bar{v}}\psi_u=\bar{y}_{\bar{u}}\phi_u+(m'\bar{y}+m''\bar{z})\psi_u=\bar{y}_{\bar{u}}\phi_u-\frac{B_{11}\bar{y}+B_{12}\bar{z}}{\sqrt{A}},\\ \bar{z}_u&=\bar{z}_{\bar{u}}\phi_u+\bar{z}_{\bar{v}}\psi_u=(n'\bar{y}+n''\bar{z})\phi_u+\bar{z}_{\bar{v}}\psi_u=\bar{z}_{\bar{v}}\psi_u+\frac{B_{21}\bar{y}+B_{22}\bar{z}}{\sqrt{A}},\end{aligned} \tag{134}$$

where we have made use of (125) and (126).

From (125) we find by differentiation

$$\begin{aligned}\bar{y}_{\bar{u}\bar{v}}&=m'\bar{y}_{\bar{u}}+(m'_{\bar{u}}+m''n')\bar{y}+(m''_{\bar{u}}+m''n'')\bar{z},\\ \bar{z}_{\bar{u}\bar{v}}&=n''\bar{z}_{\bar{v}}+(n'_{\bar{v}}+n'm')\bar{y}+(n''_{\bar{v}}+n'm'')\bar{z}.\end{aligned} \tag{135}$$

If F denotes an arbitrary function of u and v, we have

$$F_u=F_{\bar{u}}\phi_u+F_{\bar{v}}\psi_u, \qquad F_v=A_{11}F_{\bar{u}}\phi_u+A_{22}F_{\bar{v}}\psi_u, \tag{136a}$$

and

$$F_{\bar{u}}=\frac{1}{\phi_u}U(F), \qquad F_{\bar{v}}=\frac{1}{\psi_u}V(F), \tag{136b}$$

where

$$U(F)=\frac{1}{\sqrt{A}}(-A_{22}F_u+F_v), \qquad V(F)=\frac{1}{\sqrt{A}}(A_{11}F_u-F_v). \tag{136c}$$

The use of these formulas gives us the following equations for the coefficients of $\bar{y}$ and $\bar{z}$ in (135):

$$
\begin{aligned}
\phi_u \psi_u (m'_{\bar{u}} + m'' n') &= -\beta_{12}\beta_{21} + \beta_{11}\alpha_{22} - U(\beta_{11}),\\
\phi_u \psi_u (m''_{\bar{u}} + m'' n'') &= -\beta_{12}\beta_{22} + \beta_{12}\alpha_{22} - U(\beta_{12}),\\
\phi_u \psi_u (n'_{\bar{v}} + m' n') &= -\beta_{11}\beta_{21} + \beta_{21}\alpha_{11} + V(\beta_{21}),\\
\phi_u \psi_u (n''_{\bar{v}} + m'' n') &= -\beta_{12}\beta_{21} + \beta_{22}\alpha_{11} + V(\beta_{22}),
\end{aligned}
\tag{137}
$$

where

$$
\alpha_{11} = \frac{(A_{11})_u}{\sqrt{A}}, \qquad \alpha_{22} = \frac{(A_{22})_u}{\sqrt{A}}. \tag{137a}
$$

From (134) we find

$$
\begin{aligned}
\bar{y}_{uu} = \phi_u [\bar{y}_{\bar{u}\bar{u}}\phi_u + \bar{y}_{\bar{u}\bar{v}}\psi_u] + \bar{y}_{\bar{u}}\phi_{uu} - (\beta_{11})_u \bar{y} - (\beta_{12})_u \bar{z}\\
- \beta_{11}(\bar{y}_{\bar{u}}\phi_u + \bar{y}_{\bar{v}}\psi_u) - \beta_{12}(\bar{z}_{\bar{u}}\phi_u + \bar{z}_{\bar{v}}\psi_u).
\end{aligned}
$$

We substitute, into this equation, the values (125) of $\bar{y}_{\bar{v}}$ and $\bar{z}_{\bar{u}}$, and the value of $\bar{y}_{\bar{u}\bar{v}}$ from (135) and (137). The resulting expression is

$$
\begin{aligned}
\bar{y}_{uu} = \phi_u^2 \bar{y}_{\bar{u}\bar{u}} + (\phi_{uu} - 2\beta_{11}\phi_u)\bar{y}_{\bar{u}} - \beta_{12}\psi_u \bar{z}_{\bar{v}}\\
+ [\beta_{11}^2 - 2\beta_{12}\beta_{21} + \beta_{11}\alpha_{22} - (\beta_{11})_u - U(\beta_{11})]\bar{y}\\
+ [\beta_{11}\beta_{12} - 2\beta_{12}\beta_{22} + \beta_{12}\alpha_{22} - (\beta_{12})_u - U(\beta_{12})]\bar{z}.
\end{aligned}
$$

Finally we substitute this expression in (133), and transform the second equation of (133) in similar fashion. We obtain the following differential equations

$$
\begin{aligned}
\bar{y}_{\bar{u}\bar{u}} &= \bar{a}\bar{y} + \bar{b}\bar{z} + \bar{c}\bar{y}_{\bar{u}} + \bar{d}\bar{z}_{\bar{v}},\\
\bar{z}_{\bar{v}\bar{v}} &= \bar{a}'\bar{y} + \bar{b}'\bar{z} + \bar{c}'\bar{y}_{\bar{u}} + \bar{d}'\bar{z}_{\bar{v}},
\end{aligned}
\tag{138}
$$

where

$$
\begin{aligned}
\phi_u^2 \bar{a} &= -Q_{11} + \beta_{11}P_{11} - \beta_{21}P_{12} - \beta_{11}^2 + 2\beta_{12}\beta_{21}\\
&\qquad + (\beta_{11})_u - \alpha_{22}\beta_{11} + U(\beta_{11}),\\
\phi_u^2 \bar{b} &= -Q_{12} + \beta_{12}P_{11} - \beta_{22}P_{12} - \beta_{11}\beta_{12} + 2\beta_{12}\beta_{22}\\
&\qquad + (\beta_{12})_u - \alpha_{22}\beta_{12} + U(\beta_{12}),\\
\phi_u \bar{c} &= -P_{11} + 2\beta_{11} - \frac{\phi_{uu}}{\phi}, \qquad \phi_u^2 \bar{d} = \psi_u(-P_{12} + \beta_{12}),\\
\psi_u^2 \bar{a}' &= -Q_{21} + \beta_{11}P_{21} - \beta_{21}P_{22} + 2\beta_{11}\beta_{21} - \beta_{21}\beta_{22}\\
&\qquad - (\beta_{21})_u - \alpha_{11}\beta_{21} - V(\beta_{21}),\\
\psi_u^2 \bar{b}' &= -Q_{22} + \beta_{12}P_{21} - \beta_{22}P_{22} - \beta_{22}^2 + 2\beta_{12}\beta_{21}\\
&\qquad - (\beta_{22})_u - \alpha_{11}\beta_{22} - V(\beta_{22}),\\
\psi_u^2 \bar{c}' &= \phi_u(-P_{21} - \beta_{21}), \qquad \psi_u \bar{d}' = -P_{22} - 2\beta_{22} - \frac{\psi_{uu}}{\psi_u},
\end{aligned}
\tag{139}
$$

and where it is desirable to remember that

$$\sqrt{A} = A_{11} - A_{22}. \tag{140}$$

Finally, if we apply the transformation (127) to (138), we obtain the equations

$$\begin{aligned} \frac{\partial^2 \eta}{\partial \bar{u}^2} &= a\eta + b\zeta + c\frac{\partial \eta}{\partial \bar{u}} + d\frac{\partial \zeta}{\partial \bar{v}}, \\ \frac{\partial^2 \zeta}{\partial \bar{v}^2} &= a'\eta + b'\zeta + c'\frac{\partial \eta}{\partial \bar{u}} + d'\frac{\partial \zeta}{\partial \bar{v}}, \end{aligned} \tag{141}$$

where

$$\begin{aligned} a &= \bar{a} + \bar{c}\frac{\omega_{\bar{u}}}{\omega} - \frac{\omega_{\bar{u}\bar{u}}}{\omega}, \qquad b = \frac{\omega'}{\omega}\left(\bar{b} + \frac{\omega'_{\bar{v}}}{\omega'}\bar{d}\right), \\ c &= \bar{c} - 2\frac{\omega_{\bar{u}}}{\omega}, \qquad d = \frac{\omega'}{\omega}\bar{d}, \\ a' &= \frac{\omega}{\omega'}\left(\bar{a}' + \frac{\omega_{\bar{u}}}{\omega}\bar{c}'\right), \qquad b' = \bar{b}' + \frac{\omega'_{\bar{v}}}{\omega'}\bar{d}' - \frac{\omega'_{\bar{v}\bar{v}}}{\omega'}, \\ c' &= \frac{\omega}{\omega'}\bar{c}', \qquad d' = \bar{d}' - 2\frac{\omega'_{\bar{v}}}{\omega'}, \end{aligned} \tag{142}$$

and where ω and ω' satisfy the conditions (130). *Equations* (131) *and* (141) *together are the differential equations of the congruence in the canonical form,* and all of the invariants and covariants of the congruence can now be expressed in terms of the coefficients and variables of the original system (S). The functions ϕ_u, ψ_u, ω, and ω', whose actual values are required for the purpose of computing the coefficients m, n, a, b, c, d, etc., enter into the expressions for these invariants and covariants only as extraneous factors, and are eliminated entirely from the expressions for *absolute* invariants. Thus we have at once the expressions

$$\begin{aligned} m &= -\frac{\omega'}{\omega\psi_u}\beta_{12}, & n &= \frac{\omega}{\omega'\phi_u}\beta_{21}, \\ c' &= -\frac{\omega\phi_u}{\omega'\psi_u^2}(P_{21} + \beta_{21}), & d &= -\frac{\omega'\psi_u}{\omega\phi_u^2}(P_{12} - \beta_{12}), \\ W &= mn - c'd = \frac{1}{\phi_u\psi_u}(-P_{12}P_{21} + \beta_{12}P_{21} - \beta_{21}P_{12}), \end{aligned} \tag{143}$$

for the five most fundamental invariants of the congruence. If m or n vanishes one of the focal sheets degenerates into a curve; if c' or d vanishes, one of the focal sheets is a developable. If $W = 0$, the congruence is a W-congruence.

In order to find the expressions for the invariants $\overset{(\eta)}{\mathfrak{B}}$, $\overset{(\zeta)}{\mathfrak{B}}$, $\overset{(\eta)}{\mathfrak{C}}''$, and $\overset{(\zeta)}{\mathfrak{C}}''$,* the following formulas will be convenient:

* Brussels Paper, p. 20.

$$U\log\omega=\beta_{11}+\frac{\omega_u}{\omega},\qquad U\log\omega'=\beta_{22},$$

$$U\log\phi_u=\alpha_{11}+\frac{\phi_{uu}}{\phi_u},\qquad U\log\psi_u=\alpha_{22},$$

(144)

$$V\log\omega=-\beta_{11},\qquad V\log\omega'=-\beta_{22}+\frac{\omega'_u}{\omega'},$$

$$V\log\phi_u=-\alpha_{11},\qquad V\log\psi_u=-\alpha_{22}+\frac{\psi_{uu}}{\psi_u}.$$

We find

$$4\overset{(\eta)}{\mathfrak{B}}=c-\frac{1}{2}\frac{d_{\bar u}}{d}-\frac{3}{2}\frac{m_{\bar u}}{m}=\bar c-\frac{1}{2}\frac{\partial\log dm^3\omega^4}{\partial\bar u}.$$

But from (143) we have

$$dm^3\omega^4=\frac{(\omega')^4}{\phi_u^2\psi_u^2}\beta_{12}^3(P_{12}-\beta_{12}).$$

Consequently, making use of (136*b*), and (144), we find

(145*a*) $$4\phi_u\overset{(\eta)}{\mathfrak{B}}=\alpha_{11}+\alpha_{22}+2(\beta_{11}-\beta_{22})-P_{11}-\tfrac{1}{2}U\log\beta_{12}^3(P_{12}-\beta_{12}).$$

Again we have

$$4\overset{(\zeta)}{\mathfrak{B}}=c+\frac{3}{2}\frac{c'_{\bar u}}{c'}+\frac{1}{2}\frac{n_{\bar u}}{n}=\bar c+\frac{1}{2}\frac{\partial\log\dfrac{(c')^3n}{\omega^4}}{\partial\bar u},$$

and

$$\frac{(c')^3n}{\omega^4}=-\frac{\phi_u^2}{(\omega')^4\psi_u^6}\beta_{21}(P_{21}+\beta_{21})^3,$$

whence

(145*b*) $$4\phi_u\overset{(\zeta)}{\mathfrak{B}}=\alpha_{11}-3\alpha_{22}+2(\beta_{11}-\beta_{22})-P_{11}+\tfrac{1}{2}U\log\beta_{21}(P_{21}+\beta_{21})^3.$$

In the same way we find

(145*c*) $$4\psi_u\overset{(\eta)}{\mathfrak{C}}''=3\alpha_{11}-\alpha_{22}+2(\beta_{11}-\beta_{22})-P_{22}+\tfrac{1}{2}V\log\beta_{12}(P_{12}-\beta_{12})^3,$$

and

(145*d*) $$4\psi_u\overset{(\zeta)}{\mathfrak{C}}''=-\alpha_{11}-\alpha_{22}+2(\beta_{11}-\beta_{22})-P_{22}-\tfrac{1}{2}V\log\beta_{21}^3(P_{21}+\beta_{21}).$$

The values of

(146)

$$\theta^{(\eta)}=\frac{2}{\sqrt{-dm}}(m\overset{(\eta)}{\mathfrak{B}}{}^2+d\overset{(\eta)}{\mathfrak{C}}{}''^2),$$

$$\theta^{(\zeta)}=\frac{2}{\sqrt{-c'n}}(c'\overset{(\zeta)}{\mathfrak{B}}{}^2+n\overset{(\zeta)}{\mathfrak{C}}{}''^2)$$

can now easily be computed. These invariants vanish if and only if the focal sheet $S^{(\eta)}$, or $S^{(\zeta)}$, or both, are ruled surfaces.

The Laplace-Darboux invariants of the conjugate systems on S_η and S_ζ are

$$h = mn, \qquad k = mn - \frac{\partial^2 \log m}{\partial \bar{u} \partial \bar{v}},$$*

and

$$h_{-1} = mn - \frac{\partial^2 \log n}{\partial \bar{u} \partial \bar{v}}, \qquad k_{-1} = mn = h. \tag{147a}$$

We find at once

$$h = k_{-1} = -\frac{\beta_{12}\beta_{21}}{\phi_u \psi_u}. \tag{147b}$$

To obtain symmetric expressions for k and h_{-1}, we observe that, from (136b), we may compute $F_{\bar{u}\bar{v}}$ in two ways, giving

$$\phi_u \psi_u F_{\bar{u}\bar{v}} = VU(F) + \alpha_{11} U(F) = UV(F) - \alpha_{22} V(F),$$

so that we find the identity

$$VU(F) - UV(F) + \alpha_{11} U(F) + \alpha_{22} V(F) = 0, \tag{148}$$

and the symmetrical expression for $F_{\bar{u}\bar{v}}$

$$2\phi_u \psi_u F_{\bar{u}\bar{v}} = UV(F) + VU(F) + \alpha_{11} U(F) - \alpha_{22} V(F). \tag{149}$$

We propose to apply this formula for $F = \log m$ and for $F = \log n$. We find first

$$\begin{aligned}
U \log m &= -\alpha_{22} - \beta_{11} + \beta_{22} + U \log \beta_{12} - \frac{\omega_u}{\omega}, \\
V \log m &= +\alpha_{22} + \beta_{11} - \beta_{22} + V \log \beta_{12} + \frac{\omega_u'}{\omega'} - \frac{\psi_{uu}}{\psi_u}, \\
U \log n &= -\alpha_{11} + \beta_{11} - \beta_{22} + U \log \beta_{21} + \frac{\omega_u}{\omega} - \frac{\phi_{uu}}{\phi_u}, \\
V \log n &= +\alpha_{11} - \beta_{11} + \beta_{22} + V \log \beta_{21} - \frac{\omega_u'}{\omega'}.
\end{aligned} \tag{150}$$

Let us note further that

$$\begin{aligned}
U(\omega_u') &= (\beta_{22})_u \omega' + \frac{1}{2}\frac{A_u}{A} \beta_{22} \omega' + (\alpha_{22} + \beta_{22}) \omega_u', \\
U(\psi_{uu}) &= (\alpha_{22})_u \psi_u + \frac{1}{2}\frac{A_u}{A} \alpha_{22} \psi_u + 2\alpha_{22} \psi_{uu}, \\
V(\omega_u) &= -(\beta_{11})_u \omega - \frac{1}{2}\frac{A_u}{A} \beta_{11} \omega - (\alpha_{11} + \beta_{11}) \omega, \\
V(\phi_{uu}) &= -(\alpha_{11})_u \phi_u - \frac{1}{2}\frac{A_u}{A} \alpha_{11} \phi_u - 2\alpha_{11} \phi_{uu}.
\end{aligned} \tag{151}$$

* Darboux, *Théorie des surfaces*, vol. 2, and E. J. Wilczynski, *The general theory of congruences*, these Transactions, vol. 16 (1915), p. 318.

Then we find, from (149), (150), and (151),

$$
\begin{aligned}
2\phi_u \psi_u \frac{\partial^2 \log m}{\partial \bar{u} \partial \bar{v}} &= UV \log \beta_{12} + VU \log \beta_{12} + \alpha_{11} U \log \beta_{12} \\
&\quad - \alpha_{22} V \log \beta_{12} + U(\alpha_{22} + \beta_{11} - \beta_{22}) \\
&\quad - V(\alpha_{22} + \beta_{11} - \beta_{22}) + (\beta_{11} + \beta_{22} - \alpha_{22})_u \\
&\quad + \frac{1}{2} \frac{A_u}{A} (\beta_{11} + \beta_{22} - \alpha_{22}) \\
&\quad - (\alpha_{11} + \alpha_{22})(\alpha_{22} + \beta_{11} - \beta_{22}), \\
2\phi_u \psi_u \frac{\partial^2 \log n}{\partial \bar{u} \partial \bar{v}} &= UV \log \beta_{21} + VU \log \beta_{21} + \alpha_{11} U \log \beta_{21} \\
&\quad - \alpha_{22} V \log \beta_{21} + U(\alpha_{11} - \beta_{11} + \beta_{22}) \\
&\quad - V(\alpha_{11} - \beta_{11} + \beta_{22}) - (\beta_{11} + \beta_{22} - \alpha_{11})_u \\
&\quad - \frac{1}{2} \frac{A_u}{A} (\beta_{11} + \beta_{22} - \alpha_{11}) \\
&\quad - (\alpha_{11} + \alpha_{22})(\alpha_{11} - \beta_{11} + \beta_{22}).
\end{aligned} \tag{152}
$$

These expressions may be written in somewhat shorter, but less symmetric form, by making use of (148). The complete expressions for k and h_{-1} follow at once from (147) and (152).

The integrability conditions of the system, composed of (131) and (141), require the existence of a function f, of $\bar{u}$ and $\bar{v}$, such that

$$
c = \frac{\partial f}{\partial \bar{u}}, \qquad d' = \frac{\partial f}{\partial \bar{v}}. \tag{153}
$$

We wish to verify this directly and, at the same time, obtain the relation between this function and the function P, of u and v, whose partial derivatives are

$$
P_u = P_{11} + P_{22}, \qquad P_v = \Pi_{11} + \Pi_{22} - (A_{11} + A_{22})_u, \tag{154}
$$

and whose existence is assured as a result of one of the integrability conditions of our system (S), in its focal canonical form. We have

$$
c = \bar{c} - 2 \frac{\partial \log \omega}{\partial \bar{u}}, \qquad d' = \bar{d}' - 2 \frac{\partial \log \omega'}{\partial \bar{v}}.
$$

Consequently the condition $c_{\bar{v}} = d'_{\bar{u}}$, which is equivalent to (153), may be written in the form

$$
\left(\bar{c} + 2 \frac{B_{22}}{\phi_u \sqrt{A}} \right)_{\bar{v}} = \left(\bar{d}' - 2 \frac{B_{11}}{\psi_u \sqrt{A}} \right)_{\bar{u}}. \tag{155}
$$

But we find, making use of (110), (136c), (139), and (140), that

$$\bar{c} + \frac{2B_{22}}{\phi_u \sqrt{A}} = -\frac{1}{\phi_u}\left[\frac{-A_{22}P_u + P_v + (A_{11}+A_{22})_u}{\sqrt{A}} + \frac{\phi_{uu}}{\phi_u}\right]$$

$$= -P_{\bar{u}} - \frac{1}{\phi_u}\left[\frac{(A_{11}+A_{22})_u}{\sqrt{A}} + \frac{\phi_{uu}}{\phi_u}\right],$$

$$\bar{d}' - \frac{2B_{11}}{\psi_u \sqrt{A}} = -\frac{1}{\psi_u}\left[\frac{A_{11}P_u - P_v - (A_{11}+A_{22})_u}{\sqrt{A}} + \frac{\psi_{uu}}{\psi_u}\right]$$

$$= -P_{\bar{v}} - \frac{1}{\psi_u}\left[\frac{-(A_{11}+A_{22})_u}{\sqrt{A}} + \frac{\psi_{uu}}{\psi_u}\right].$$

These expressions will satisfy (155) if and only if there exists a function F whose partial derivatives with respect to $\bar{u}$ and $\bar{v}$ can be identified with the bracket terms of the two right members. We must have in that case

$$-A_{22}F_u + F_v = (A_{11}+A_{22})_u + \frac{\phi_{uu}}{\phi_u}(A_{11}-A_{22}),$$

$$+A_{11}F_u - F_v = -(A_{11}+A_{22})_u + \frac{\psi_{uu}}{\psi_u}(A_{11}-A_{22}),$$

whence

$$F_u = \frac{\phi_{uu}}{\phi_u} + \frac{\psi_{uu}}{\psi_u}, \qquad F_v = (A_{11}+A_{22})_u + A_{11}\frac{\phi_{uu}}{\phi_u} + A_{22}\frac{\psi_{uu}}{\psi_u}.$$

The consistency of these conditions may be verified directly as follows.

The first equation gives on integration

$$F = \log \phi_u \psi_u + V(v)$$

where $V(v)$ is a function of v alone. If we substitute this expression for F into the equation for F_v, and make use of the conditions

$$\phi_v = A_{11}\phi_u, \qquad \psi_v = A_{22}\psi_u$$

which define ϕ and ψ, we find $V' = 0$. Consequently we have found

$$\bar{c} + \frac{2B_{22}}{\phi_u \sqrt{A}} = -\frac{\partial}{\partial \bar{u}}(P + \log C\phi_u \psi_u),$$

$$\bar{d}' - \frac{2B_{11}}{\psi_u \sqrt{A}} = -\frac{\partial}{\partial \bar{v}}(P + \log C\phi_u \psi_u),$$

where C is a non-vanishing constant, and therefore

$$c = -\frac{\partial}{\partial \bar{u}}(P + \log C\phi_u \psi_u + 2\log \omega' + 2\log \omega) = \frac{\partial f}{\partial \bar{u}},$$

$$d' = -\frac{\partial}{\partial \bar{v}}(P + \log C\phi_u \psi_u + 2\log \omega + 2\log \omega') = \frac{\partial f}{\partial \bar{v}},$$

whence

$$f = -[P + \log(\omega\omega')^2 \phi_u \psi_u + k]$$

where k is a constant. According to (105) and (154) we have

$$P_u = p_u - \frac{A_u}{A}, \qquad P_v = p_v - \frac{A_v}{A}$$

and therefore

(156) $$f = -[p + \log A^{-1} \phi_u \psi_u (\omega\omega')^2 + k]$$

where p is defined by the consistent conditions

$$p_u = p_{11} + p_{22}, \qquad p_v = \pi_{11} + \pi_{22} - (a_{11} + a_{22})_u,$$

and where p is closely connected with the value of

$$D = |y_u, z_u, y, z|,$$

the principal determinant of system (S), this fourth order determinant being formed from four independent solutions of (S).

By using the differential equations of (S), we find

$$D_u = |y_{uu}, z_u, y, z| + |y_u, z_{uu}, y, z| = -(p_{11} + p_{22}) D$$

$$D_v = |y_{uv}, z_u, y, z| + |y, z_{uv}, y, z| + |y_u, z_u, y_v, z| + |y_u, z_u, y, z_v|$$

$$= (b_{11} + b_{22} + c_{11} + c_{22}) D_v = -[\pi_{11} + \pi_{22} - (a_{11} + a_{22})_u] D,$$

so that

(157) $$p_u = p_{11} + p_{22} = -\frac{D_u}{D}, \qquad p_v = \pi_{11} + \pi_{22} - (a_{11} + a_{22})_u = -\frac{D_v}{D}.$$

Consequently we must have

(158) $$D = Ce^{-p}$$

where C is a non-vanishing constant whose actual value will depend upon what particular solutions of (S) have been used in forming the determinant D. We may write finally

(159) $$f = \log AD\phi_u^{-1} \psi_u^{-1} (\omega\omega')^{-2} K$$

where K is a non-vanishing constant.

On the other hand we have also

(160) $$f = \log K' \Delta^*$$

where the fourth order determinant

(161) $$\Delta = |\eta, \zeta, \eta_{\bar{u}}, \zeta_{\bar{v}}|$$

is formed from the corresponding four solutions of the system (131), (141). The resulting relation between D and Δ may be verified directly.

* Brussels Paper, p. 26.

12. The differential equations of the net of ruled surfaces in planar coördinates and the principle of duality

Let us return to the differential equations (S) of the one parameter family of ruled surfaces. The generating lines of these surfaces have so far been regarded as the lines of junction of corresponding points, P_y and P_z, of the two surfaces of reference S_y and S_z. We propose now to think of these same lines as lines of intersections of corresponding planes.

Let R be a ruled surface of the family $v = \text{const.}$ The coördinates of the plane tangent to R at P_y are the four third order determinants of the form

$$r = |y, z, y_u|. \tag{162a}$$

Similarly the determinants of the form

$$s = |y, z, z_u| \tag{162b}$$

serve as coördinates for the plane tangent to R at P_z. We propose to obtain a system of differential equations similar to (S) which will be satisfied by the four pairs of functions (r_i, s_i), $(i = 1, 2, 3, 4)$, thus obtaining a representation for the net of ruled surfaces in planar coördinates.

For purposes of abbreviation let us write

$$t = |y, z_u, y_u|, \qquad t' = |z, z_u, y_u|. \tag{163}$$

We find from (162a) and (162b), by differentiation, making use of the equations of system (S), equations (6), and familiar theorems about determinants,

$$\begin{aligned} r_u &= -p_{11} r - p_{12} s + t, \\ r_v &= (b_{11} + b_{22} + c_{11}) r + c_{12} s - a_{12} t' + a_{22} t, \\ s_u &= -p_{21} r - p_{22} s + t', \\ s_v &= c_{21} r + (b_{11} + b_{22} + c_{22}) s + a_{11} t' - a_{21} t, \end{aligned} \tag{164}$$

whence follows, by elimination of t and t',

$$\begin{aligned} r_v = a_{22} r_u - a_{12} s_u + (b_{11} + b_{22} + c_{11} - a_{12} p_{21} + a_{22} p_{11}) r \\ + (c_{12} - a_{12} p_{22} + a_{22} p_{12}) s, \\ s_v = -a_{21} r_u + a_{11} s_u + (c_{21} + a_{11} p_{21} - a_{21} p_{11}) r \\ + (b_{11} + b_{22} + c_{22} + a_{11} p_{22} - a_{21} p_{12}) s. \end{aligned} \tag{165}$$

Let us put

$$\mathfrak{y} = rD^{-1/2}, \qquad \mathfrak{z} = sD^{-1/2} \tag{166}$$

where D is the fourth-order determinant

$$D = |y_u, z_u, y, z|, \tag{167}$$

whose partial derivatives are given in (157). The resulting system of differential equations may be written

$$
\begin{aligned}
&\mathfrak{y}_{uu} + \mathfrak{p}_{11}\,\mathfrak{y}_u + \mathfrak{p}_{12}\,\mathfrak{z}_u + \mathfrak{q}_{11}\,\mathfrak{y} + \mathfrak{q}_{12}\,\mathfrak{z} = 0,\\
&\mathfrak{z}_{uu} + \mathfrak{p}_{21}\,\mathfrak{y}_u + \mathfrak{p}_{22}\,\mathfrak{z}_u + \mathfrak{q}_{21}\,\mathfrak{y} + \mathfrak{q}_{22}\,\mathfrak{z} = 0,\\
&\mathfrak{y}_v = \mathfrak{a}_{11}\,\mathfrak{y}_u + \mathfrak{a}_{12}\,\mathfrak{z}_u + \mathfrak{b}_{11}\,\mathfrak{y} + \mathfrak{b}_{12}\,\mathfrak{z},\\
&\mathfrak{z}_v = \mathfrak{a}_{21}\,\mathfrak{y}_u + \mathfrak{a}_{22}\,\mathfrak{z}_u + \mathfrak{b}_{21}\,\mathfrak{y} + \mathfrak{b}_{22}\,\mathfrak{z}.
\end{aligned}
\tag{168}
$$

The values of the coefficients $\mathfrak{p}_{ik}$ and $\mathfrak{q}_{ik}$ may be transcribed from the theory of ruled surfaces.*

The values of $\mathfrak{a}_{ik}$ and $\mathfrak{b}_{ik}$ are easily computed from (165) and (166). We find

$$
\begin{aligned}
&\mathfrak{a}_{11} = a_{22}, \quad \mathfrak{a}_{12} = -a_{12}, \quad \mathfrak{a}_{21} = -a_{21}, \quad \mathfrak{a}_{22} = a_{11},\\
&\mathfrak{b}_{11} = b_{11} + \tfrac{1}{2}(a_{11} - a_{22})_u - \tfrac{1}{2}(a_{11} - a_{22})\,p_{11} + \tfrac{1}{2}\,a_{21}\,p_{12} - a_{12}\,p_{21},\\
&\mathfrak{b}_{12} = b_{12} + (a_{12})_u - (a_{11} - a_{22})\,p_{12} + \tfrac{1}{2}\,a_{12}(p_{11} - p_{22}) - a_{12}\,p_{22},\\
&\mathfrak{b}_{21} = b_{21} + (a_{21})_u + (a_{11} - a_{22})\,p_{21} - \tfrac{1}{2}\,a_{21}(p_{11} - p_{22}) - a_{21}\,p_{11},\\
&\mathfrak{b}_{22} = b_{22} - \tfrac{1}{2}(a_{11} - a_{22})_u + \tfrac{1}{2}(a_{11} - a_{22})\,p_{22} + \tfrac{1}{2}\,a_{12}\,p_{21} - a_{21}\,p_{12},\\
&\mathfrak{p}_{ik} = p_{ik},\\
&\mathfrak{q}_{11} = q_{11} + \tfrac{1}{4}(u_{11} - u_{22}), \quad \mathfrak{q}_{12} = q_{12} + \tfrac{1}{2}\,u_{12},\\
&\mathfrak{q}_{21} = q_{21} + \tfrac{1}{2}\,u_{21}, \quad \mathfrak{q}_{22} = q_{22} - \tfrac{1}{4}(u_{11} - u_{22}).
\end{aligned}
\tag{169}
$$

It is an easy matter to verify the complete reciprocity between systems (S) and (168). We shall speak of them as systems *adjoint* to each other. In our interpretation the two systems correspond to the same congruence but in dualized representation. Clearly, if we re-interpret $\mathfrak{y}_1 \cdots \mathfrak{y}_4$ and $\mathfrak{z}_1, \cdots, \mathfrak{z}_4$ as point-coördinates, we obtain instead two nets of ruled surfaces which are dual to each other. Consequently those properties of a net whose analytic expressions remain unaltered by the transformations (169) are dualistic properties.

13. Conjugate systems of ruled surfaces

Let us put

$$
\rho^{(u)} = 2y_u + p_{11}\,y + p_{12}\,z, \qquad \sigma^{(u)} = 2z_u + p_{21}\,y + p_{22}\,z. \tag{170}
$$

These expressions determine two points, $\rho^{(u)}$ and $\sigma^{(u)}$, such that the line joining them is a generator, of the same set as $P_y P_z$, on the quadric surface H which osculates the ruled surface $v = \text{const.}$ along $P_y P_z$. The lines which

* *Proj. Diff. Geom.*, p. 137.

join y to $\rho^{(u)}$, and z to $\sigma^{(u)}$ are generators of H of the second kind. Referred to the tetrahedron of these four points, and an appropriately chosen unit-point, the equation of H is

$$x_1 x_4 - x_2 x_3 = 0.* \tag{171}$$

We introduce two other points, by means of the expressions,

$$\rho^{(v)} = 2y_v + r_{11} y + r_{12} z, \qquad \sigma^{(v)} = 2z_v + r_{21} y + r_{22} z, \tag{172}$$

where the coefficients r_{ik} are defined by (15). These points are on the quadric H', which osculates the ruled surface $u = \text{const.}$ along $P_y P_z$, and their position on this quadric is determined by P_y and P_z by the same construction which, executed on H, gives rise to $\rho^{(u)}$ and $\sigma^{(u)}$.

Let us study the quadric H'. An arbitrary point on H' is given by an expression of the form

$$\beta_1(\alpha_1 y + \alpha_2 z) + \beta_2(\alpha_1 \rho^{(v)} + \alpha_2 \sigma^{(v)}). \tag{173}$$

But we have

$$\rho^{(v)} = 2(a_{11} y_u + a_{12} z_u + b_{11} y + b_{12} z) + r_{11} y + r_{12} z,$$

$$\sigma^{(v)} = 2(a_{21} y_u + a_{22} z_u + b_{21} y + b_{22} z) + r_{21} y + r_{22} z,$$

and

$$2y_u = \rho^{(u)} - p_{11} y - p_{12} z, \qquad 2z_u = \sigma^{(u)} - p_{21} y - p_{22} z,$$

whence

$$\rho^{(v)} = a_{11}\rho^{(u)} + a_{12}\sigma^{(u)} + (2b_{11} + r_{11} - a_{11}p_{11} - a_{12}p_{21})y + (2b_{12} + r_{12} - a_{11}p_{12} - a_{12}p_{22})z,$$

$$\sigma^{(v)} = a_{21}\rho^{(u)} + a_{22}\sigma^{(u)} + (2b_{21} + r_{21} - a_{21}p_{11} - a_{22}p_{21})y + (2b_{22} + r_{22} - a_{21}p_{12} - a_{22}p_{22})z,$$

or, on account of (37),

$$\begin{aligned} \rho^{(v)} &= a_{11}\rho^{(u)} + a_{12}\sigma^{(u)} + (r_{11} - \pi_{11})y + (r_{12} - \pi_{12})z, \\ \sigma^{(v)} &= a_{21}\rho^{(u)} + a_{22}\sigma^{(u)} + (r_{21} - \pi_{21})y + (r_{22} - \pi_{22})z. \end{aligned} \tag{174}$$

If we introduce these expressions into (173) we obtain an expression of the form

$$x_1 y + x_2 z + x_3 \rho^{(u)} + x_4 \sigma^{(u)},$$

where

$$\begin{aligned} x_1 &= \beta_1\alpha_1 + \beta_2\alpha_1(r_{11} - \pi_{11}) + \beta_2\alpha_2(r_{21} - \pi_{21}), \\ x_2 &= \beta_1\alpha_2 + \beta_2\alpha_1(r_{12} - \pi_{12}) + \beta_2\alpha_2(r_{22} - \pi_{22}), \\ x_3 &= \beta_2\alpha_1 a_{11} + \beta_2\alpha_2 a_{21}, \\ x_4 &= \beta_2\alpha_1 a_{12} + \beta_2\alpha_2 a_{22}, \end{aligned} \tag{175}$$

* *Proj. Diff. Geom.*, p. 191.

and where we may regard x_1, x_2, x_3, x_4 as the homogeneous coördinates of an arbitrary point of H' referred to the tetrahedron of the four points y, z, $\rho^{(u)}$, $\sigma^{(u)}$. To obtain the equation of H' we eliminate $\alpha_1, \beta_1, \alpha_2, \beta_2$ from (175). From the last two equations we find

$$a_2 \beta_2 \alpha_1 = a_{22} x_3 - a_{21} x_4, \qquad a_2 \beta_2 \alpha_2 = - a_{12} x_3 + a_{11} x_4$$

where we have put $a_2 = a_{11} a_{22} - a_{12} a_{21}$ as before. Substitution of these values into the first two equations of (175) gives

$$\begin{aligned} a_2 \alpha_1 \beta_1 &= a_2 x_1 - (r_{11} - \pi_{11})(a_{22} x_3 - a_{21} x_4) \\ &\qquad - (r_{21} - \pi_{21})(- a_{12} x_3 + a_{11} x_4), \\ a_2 \beta_1 \alpha_2 &= a_2 x_2 - (r_{12} - \pi_{12})(a_{22} x_3 - a_{21} x_4) \\ &\qquad - (r_{22} - \pi_{22})(- a_{12} x_3 + a_{11} x_4), \end{aligned}$$

whence finally

$$\begin{aligned} &[a_2 x_1 + (\pi_{11} - r_{11})(a_{22} x_3 - a_{21} x_4) \\ &\qquad + (\pi_{21} - r_{21})(- a_{12} x_3 + a_{11} x_4)](- a_{12} x_3 + a_{11} x_4) \\ &- [a_2 x_2 + (\pi_{12} - r_{12})(a_{22} x_3 - a_{21} x_4) \\ &\qquad + (\pi_{22} - r_{22})(- a_{12} x_3 + a_{11} x_4)](a_{22} x_3 - a_{21} x_4) = 0 \end{aligned} \tag{176}$$

the equation of H'.

Let us consider any point P' of the line $P_y P_z$. Let the coördinates of such a point be x_1', x_2', 0, 0. The plane tangent to H' at P' will have the equation

$$- (a_{12} x_1' + a_{22} x_2') x_3 + (a_{11} x_1' + a_{21} x_2') x_4 = 0, \tag{177}$$

if we assume $a_2 \neq 0$. The corresponding tangent plane of H is given by

$$- x_2' x_3 + x_1' x_4 = 0. \tag{178}$$

These two planes will coincide, if and only if

$$a_{12} (x_1')^2 - (a_{11} - a_{22}) x_1' x_2' - a_{21} (x_2')^2 = 0,$$

that is, if P' is one of the two foci of the line $P_y P_z$.

Thus the two quadrics H and H' are tangent to each other at the foci. The two planes of contact are given by

$$a_{12} x_3^2 - (a_{11} - a_{22}) x_3 x_4 - a_{21} x_4^2 = 0. \tag{179}$$

Since all of the ruled surfaces of the net must be tangent to the focal sheets, it is clear that these two planes are the planes tangent to the focal sheets of the congruence, and we shall call them the focal planes of the line $P_y P_z$.

Let us write equation (178) in the form

$$x_3 - \lambda x_4 = 0,$$

where $\lambda = x_1' : x_2'$. This plane and the plane

$$x_3 - \mu x_4 = 0$$

will form a pair, given by

$$x_3^2 - (\lambda + \mu) x_3 x_4 + \lambda\mu x_4^2 = 0$$

which is divided harmonically by the focal planes (179), if and only if

(180) $$2a_{12}\lambda\mu - (a_{11} - a_{22})(\lambda + \mu) - 2a_{21} = 0,$$

whence

$$\mu = \frac{2a_{21} + (a_{11} - a_{22})\lambda}{-(a_{11} - a_{22}) + 2a_{12}\lambda}.$$

The plane (177) will coincide with the plane $x_3 - \mu x_4 = 0$ for all values of λ, if and only if the equation

$$[(a_{11} - a_{22})\lambda + 2a_{21}](a_{12}\lambda + a_{22}) = [2a_{12}\lambda - (a_{11} - a_{22})](a_{11}\lambda + a_{21})$$

is an identity, that is, if

$$a_{11}^2 = a_{22}^2, \qquad (a_{11} - a_{22})a_{12} = 2a_{11}a_{12}, \qquad 2a_{21}a_{22} = -a_{21}(a_{11} - a_{22}).$$

If a_{12} and a_{21} are not both equal to zero, we conclude

$$a_{11} + a_{22} = 0.$$

If a_{12} and a_{21} are both zero, these conditions are satisfied also by

$$a_{12} = a_{21} = a_{11} - a_{22} = 0;$$

but in this case the congruence degenerates into a single ruled surface.

Let us agree to speak of a net of ruled surfaces as a *conjugate net* if the osculating quadrics, H and H', of the two ruled surfaces u = const. and v = const. which meet along a line $P_y P_z$ of the congruence, are so related that the planes tangent to H and H' respectively at every point P of $P_y P_z$ are divided harmonically by the focal planes of $P_y P_z$. We have just shown that *the net of ruled surfaces defined by system* (S) *is a conjugate net, if and only if*

(181) $$a_1 = a_{11} + a_{22} = 0,$$

provided that (S) defines a proper net of ruled surfaces which does not degenerate into a single ruled surface.

Thus the method of Article 8 for the construction of invariants of a one-parameter family of ruled surfaces consists geometrically of associating with the given one-parameter family of ruled surfaces its conjugate family; the

invariants of the conjugate net obtained in this way are at the same time the invariants of the given one-parameter family.

Clearly the developables of the congruence are their own conjugates. Thus, making use of this new notion of conjugate systems of ruled surfaces, the analogy between the developables of a congruence and the asymptotic lines on a surface becomes very evident.

Since the quadrics H and H' have a line of the congruence in common, the rest of their intersection will be, in general, a twisted cubic. We shall refrain, at present, from any discussion of the congruence of cubics which arises in this way.

14. Geometric significance of some of the invariants obtained previously

The factors of the covariant

$$a_{21}\,y^2 - (a_{11} - a_{22})\,yz - a_{12}\,z^2 \tag{182}$$

determine the foci of the line $P_y P_z$. The factors of

$$u_{21}\,y^2 - (u_{11} - u_{22})\,yz - u_{12}\,z^2 \tag{183}$$

determine the flecnodes of $P_y P_z$ when $P_y P_z$ is regarded as a generator of a ruled surface $v = \text{const.}$ The factors of

$$\begin{aligned}[(u_{11} - u_{22})\,v_{21} - (v_{11} - v_{22})\,u_{21}]\,y^2 + 2\,(u_{12}\,v_{21} - u_{21}\,v_{12})\,yz \\ + [(u_{11} - u_{22})\,v_{12} - (v_{11} - v_{22})\,u_{12}]\,z^2\end{aligned} \tag{184}$$

determine the complex points of $P_y P_z$, if again $P_y P_z$ be regarded as a generator of a ruled surface of the family $v = \text{const.}$*

The discriminants of these three quadratic covariants are A, $\theta_{4,\,0}$, and $4\theta_{10,\,0}$, and the significance of the vanishing of these invariants is therefore apparent.

The bilinear invariant of (182) and (183) is

$$-\tfrac{1}{2}(a_{11} - a_{22})(u_{11} - u_{22}) - a_{12}\,u_{21} - a_{21}\,u_{12} = \theta_{1,\,1},$$

according to (59). Therefore, *the flecnodes of a generator of a ruled surface $v = const.$, and the foci of that line separate each other harmonically, if and only if $\theta_{1,\,1} = 0$.*

The bilinear invariant of (182) and (184) is

$$\begin{aligned}a_{21}[(u_{11} - u_{22})\,v_{12} - (v_{11} - v_{22})\,u_{12}] - a_{12}[(u_{11} - u_{22})\,v_{21} - (v_{11} - v_{22})\,u_{21}] \\ + (a_{11} - a_{22})(u_{12}\,v_{21} - u_{21}\,v_{12}),\end{aligned}$$

and this is equal to

* *Proj. Diff. Geom.*, p. 208.

$$\begin{vmatrix} a_{11} - a_{22}, & a_{12}, & a_{21} \\ u_{11} - u_{22}, & u_{12}, & u_{21} \\ v_{11} - v_{22}, & v_{12}, & v_{21} \end{vmatrix} = (a, u, v) = \theta_{4,1}$$

according to (60).

Consequently *the complex points of a generator of a ruled surface $v = const.$, and the foci of the same line separate each other harmonically if and only if* $\theta_{4,1} = 0$.

The bilinear invariant of (182) and (184) vanishes identically, corresponding to the fact that the flecnodes and complex points always separate each other harmonically. The quantity

$$\frac{\theta_{1,1}^2}{A\theta_{4,0}} \tag{185}$$

is an absolute invariant of the one-parameter family of ruled surfaces which determines the cross ratio α of the flecnodes and foci by means of the equation

$$\theta_{1,1}^2(\alpha - 1)^2 - A\theta_{4,0}(\alpha + 1)^2 = 0. \tag{186}$$

The form of (184) suggests a new covariant

$$\begin{aligned}[(u_{11} - u_{22})a_{21} - (a_{11} - a_{22})u_{21}]y^2 + 2(u_{12}a_{21} - u_{21}a_{12})yz \\ + [(u_{11} - u_{22})a_{12} - (a_{11} - a_{22})u_{12}]z^2.\end{aligned} \tag{187}$$

Clearly the bilinear invariant of (182) and (187) is $(a, u, a) = 0$, and that of (183) and (187) is $(a, u, u) = 0$. Consequently (187) represents the pair of points on $P_y P_z$ which are harmonic conjugates of each other both with respect to the flecnodes and the foci of $P_y P_z$. The discriminant of (187) is a new invariant of the one-parameter family of ruled surfaces.

15. The Laplace transformation

The first Laplace transformation is given by

$$\eta_1 = \eta_{\bar{u}} - \frac{m_{\bar{u}}}{m}\eta = m\frac{\partial}{\partial \bar{u}}\left(\frac{\eta}{m}\right), \qquad \zeta_1 = \frac{\eta}{m}. \tag{188}$$

We wish to express these covariants in terms of the variables and coefficients of system (S). We find, from (188) and (136),

$$\eta_1 = -\frac{\omega'\beta_{12}}{\omega\psi_u}\frac{\partial}{\partial \bar{u}}\left(\frac{-\bar{y}\psi_u}{\omega'\beta_{12}}\right) = \frac{\omega'\beta_{12}}{\omega\phi_u\psi_u}U\left(\frac{\psi_u\bar{y}}{\omega'\beta_{12}}\right).$$

But we have found

$$U(\psi_u) = \alpha_{22}\psi_u, \qquad U(\omega') = \beta_{22}\omega',$$

so that we find

$$\omega\phi_u\eta_1 = \beta_{12}U\left(\frac{\bar{y}}{\beta_{12}}\right) + (\alpha_{22} - \beta_{22})\bar{y}, \tag{189}$$

where $\bar{y}$ is given by (98), and

$$\zeta_1 = -\frac{\psi_u \bar{y}}{\omega' \beta_{12}}. \tag{190}$$

In the same way we find for the minus first Laplace transformation

$$\eta_{-1} = \frac{\phi_u \bar{z}}{\omega \beta_{21}}, \qquad \omega' \psi_u \zeta_{-1} = \beta_{21} V\left(\frac{\bar{z}}{\beta_{21}}\right) - (\alpha_{11} - \beta_{11})\bar{z}, \tag{191}$$

where the value of $\bar{z}$ is also given by (98).

The invariants h_{-1} and k_{-1} have been calculated before. (See equations (147) to (152)), and we have

$$h_1 = k = mn, \qquad k_1 = m_1 n_1 - \frac{\partial^2 \log m_1}{\partial \bar{u} \partial \bar{v}} = k - \frac{\partial^2 \log km}{\partial \bar{u} \partial \bar{v}}, \tag{192}$$

where, on account of (149),

$$2\phi_u \psi_u \frac{\partial^2 \log km}{\partial \bar{u} \partial \bar{v}} = UV(km) + VU(km) + \alpha_{11} U(km) - \alpha_{22} V(km). \tag{193}$$

PALMER LAKE, COLO.,
December 3, 1919.

NETS OF SPACE CURVES*

BY

G. M. GREEN

In the metric differential geometry of a curved surface, the discussion is often centered about the gaussian parameter form of representation, in which the points on the surface are fixed by an arbitrary system of curvilinear coördinates. So far as we are aware, the configuration of two arbitrary families of curves on a surface, i.e., an arbitrary net or system of curvilinear coördinates, has not been studied for its own sake. By this we mean that the properties usually investigated belong to the surface, and not intrinsically to the net itself.

It is our purpose in the present paper to enter upon this apparently neglected field, by studying projectively an arbitrary net of space curves. The point of departure is a completely integrable system of partial differential equations, any fundamental system of solutions of which defines the geometric configuration, and which has moreover the property that the similar configuration defined by any other fundamental system of solutions is a projective transformation of the first.† The projective properties of the net will therefore be expressed in terms of the differential equations. Of course all properties belonging to the surface on which the net lies belong also to the net itself, but we shall find that there are in addition many others peculiar to the net.

1. The differential equations of the problem

Suppose the homogeneous coördinates $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$ of a point in space to be given as analytic functions of two variables:

* Presented to the Society, April 24, 1915.

The manuscript of this paper was found among the papers in the late Dr. Green's handwriting which were turned over to me for investigation. It was complete except for a few references, which I have supplied; the only changes which I have permitted myself to make are concerned with two or three places where the language seemed to be ambiguous. This paper constitutes a notable addition to Green's writings, and shows once more the fertility of his imagination and his unfailing ability to see something new even in the most familiar fields.—E. J. Wilczynski.

† The first to use systematically completely integrable systems of this kind in projective differential geometry was Wilczynski. See for instance his five memoirs on the theory of surfaces, these Transactions (1907–1910).

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4). \tag{1}$$

For v = const., these equations define a one-parameter family of curves C_u, and for u = const., a one-parameter family of curves C_v; the net formed by the curves C_u and C_v lies on a surface S_y.

In our subsequent analysis we shall suppose that the curves C_u are not conjugate to the curves C_v. In other words, the four functions (1) are not all solutions of an equation of the Laplace type:*

$$\alpha y_{uv} + \beta y_u + \gamma y_v + \delta y = 0. \tag{2}$$

We may express this otherwise, by saying that the determinant

$$W = \begin{vmatrix} y_{uv}^{(1)} & y_u^{(1)} & y_v^{(1)} & y^{(1)} \\ y_{uv}^{(2)} & y_u^{(2)} & y_v^{(2)} & y^{(2)} \\ y_{uv}^{(3)} & y_u^{(3)} & y_v^{(3)} & y^{(3)} \\ y_{uv}^{(4)} & y_u^{(4)} & y_v^{(4)} & y^{(4)} \end{vmatrix} \tag{3}$$

is not zero. Geometrically, this means that for values of u and v corresponding to points in the region of space for which $W \neq 0$, the four points $y_{uv}^{(k)}$, $y_u^{(k)}$, $y_v^{(k)}$, $y^{(k)}$, $(k = 1, 2, 3, 4)$ are not coplanar, so that any other point of space must be linearly dependent on these four. In particular, this must be the case with the two points $y_{uu}^{(k)}$ and $y_{vv}^{(k)}$, so that relations of the form

$$\begin{aligned} y_{uu}^{(k)} &= a y_{uv}^{(k)} + b y_u^{(k)} + c y_v^{(k)} + d y^{(k)}, \\ y_{vv}^{(k)} &= a' y_{uv}^{(k)} + b' y_u^{(k)} + c' y_v^{(k)} + d' y^{(k)} \end{aligned} \qquad (k = 1, 2, 3, 4) \tag{4}$$

must exist, where the coefficients a, b, etc., are functions of u and v. *The functions* (1) *are solutions of the partial differential equations*

$$\begin{aligned} y_{uu} &= a y_{uv} + b y_u + c y_v + d y, \\ y_{vv} &= a' y_{uv} + b' y_u + c' y_v + d' y. \end{aligned} \tag{5}$$

In fact, we may determine the coefficients in these differential equations by substituting successively $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$ for y, thus obtaining the eight equations (4), and then solving these for a, b, etc. This is possible, since we have supposed the determinant W given by (3) to be different from zero.

Similarly, all derivatives of order higher than the second are, like y_{uu} and y_{vv}, expressible linearly in terms of y_{uv}, y_u, y_v, y; in fact, these expressions

* Darboux, *Théorie générale des surfaces*, vol. 1, p. 122. Conjugate nets, as is well known, have received a good deal of attention, especially because of their intimate relation to the theory of congruences. For a discussion of the subject from the point of view of the present paper, see G. M. Green, *Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface*, (First Memoir), American Journal of Mathematics, vol. 37 (1915), pp. 215–246.

may actually be found by repeated differentiation of equations (5). It is apparent, then, that system (5) is completely integrable, and has the functions $y^{(k)}$ $(k = 1, 2, 3, 4)$ as a fundamental set of solutions. Moreover, any other fundamental set of solutions $\bar{y}^{(k)}$—i.e., a set for which, as for $y^{(k)}$, the determinant $|\bar{y}_{uv}, \bar{y}_u, \bar{y}_v, \bar{y}|$ is different from zero—is expressible linearly, with constant coefficients, in terms of the original set $y^{(k)}$:

$$(6)\quad \bar{y}^{(k)} = c_{k1}\, y^{(1)} + c_{k2}\, y^{(2)} + c_{k3}\, y^{(3)} + c_{k4}\, y^{(4)}, \qquad |c_{kl}| \neq 0 \quad (k = 1, 2, 3, 4).$$

In other words, *the net defined by any fundamental system of solutions of equations* (5) *is a projective transformation of the net defined by any other fundamental system.* Consequently, any geometric property of the net, expressible in terms of the differential equations, is common to the net and all of its projective transformations; in other words, the property is projective.

We shall base our study entirely upon the system of differential equations; evidently, if a system of the form of equations (5) be written down at random, it will be incompatible, so that relations must exist among the coefficients in order that the system be completely integrable and have the properties just described. We consider therefore this question. By differentiation of equations (5) we obtain the four equations

$$\begin{aligned}
y_{uuu} - a y_{uuv} &= (a_u + ab + c)\, y_{uv} + (b^2 + b_u + d)\, y_u \\
&\qquad + (bc + c_u)\, y_v + (bd + d_u)\, y, \\
y_{uuv} - a y_{uvv} &= (a_v + b + ca')\, y_{uv} + (b_v + cb')\, y_u \\
&\qquad + (cc' + c_v + d)\, y_v + (cd' + d_v)\, y, \\
y_{uvv} - a' y_{uuv} &= (a'_u + b'a + c')\, y_{uv} + (b'b + b'_u + d')\, y_u \\
&\qquad + (b'c + c'_u)\, y_v + (b'd + d'_u)\, y, \\
y_{vvv} - a' y_{uvv} &= (a'_v + b' + c'a')\, y_{uv} + (b'_v + c'b')\, y_u \\
&\qquad + (c'^2 + c'_v + d')\, y_v + (c'd' + d'_v)\, y,
\end{aligned} \tag{7}$$

in which use has been made of equations (5) to replace y_{uu} and y_{vv} by their values in terms of y_{uv}, y_u, y_v, y. From the second and third of equations (7) may be obtained the values for the derivatives y_{uuv} and y_{uvv}, and then these may be used to find y_{uuu} and y_{vvv} from the first and fourth of equations (7). In this way we obtain for the four derivatives of the third order the expressions

$$\begin{aligned}
y_{uuu} &= a^{(30)}\, y_{uv} + b^{(30)}\, y_u + c^{(30)}\, y_v + d^{(30)}\, y, \\
y_{uuv} &= a^{(21)}\, y_{uv} + b^{(21)}\, y_u + c^{(21)}\, y_v + d^{(21)}\, y, \\
y_{uvv} &= a^{(12)}\, y_{uv} + b^{(12)}\, y_u + c^{(12)}\, y_v + d^{(12)}\, y, \\
y_{vvv} &= a^{(03)}\, y_{uv} + b^{(03)}\, y_u + c^{(03)}\, y_v + d^{(03)}\, y,
\end{aligned} \tag{8}$$

where

$$
\begin{aligned}
a^{(21)} &= [a_v + b + ca' + a(a'_u + b'a + c')]/(1 - aa'),\\
b^{(21)} &= [b_v + cb' + a(b'b + b'_u + d')]/(1 - aa'),\\
c^{(21)} &= [cc' + c_v + d + a(b'c + c'_u)]/(1 - aa'),\\
d^{(21)} &= [cd' + d_v + a(b'd + d'_u)]/(1 - aa'),\\
a^{(12)} &= [a'(a_v + b + ca') + a'_u + b'a + c']/(1 - aa'),\\
b^{(12)} &= [a'(b_v + cb') + b'b + b'_u + d']/(1 - aa'),\\
c^{(12)} &= [a'(cc' + c_v + d) + b'c + c'_u]/(1 - aa'),\\
d^{(12)} &= [a'(cd' + d_v) + b'd + d'_u]/(1 - aa'),
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
a^{(30)} &= aa^{(21)} + a_u + ab + c, & b^{(30)} &= ab^{(21)} + b^2 + b_u + d,\\
c^{(30)} &= ac^{(21)} + bc + c_u, & d^{(30)} &= ad^{(21)} + bd + d_u,\\
a^{(03)} &= a'a^{(12)} + a'_v + b' + c'a', & b^{(03)} &= a'b^{(12)} + b'_v + c'b',\\
c^{(03)} &= a'c^{(12)} + c'^2_v + c'_v + d', & d^{(03)} &= a'd^{(12)} + c'd' + d'_v.
\end{aligned}
$$

We suppose, of course, that

$$1 - aa' \neq 0, \tag{10}$$

which is tantamount to assuming that the surface is not developable.*

The derivatives of the fourth order are likewise expressible linearly in terms of y_{uv}, y_u, y_v, y, but not all of them uniquely. There are in fact three possibilities for ambiguous expressions, since

$$\frac{\partial y_{uuu}}{\partial v} = \frac{\partial y_{uuv}}{\partial u}, \qquad \frac{\partial y_{uuv}}{\partial v} = \frac{\partial y_{uvv}}{\partial u}, \qquad \frac{\partial y_{uvv}}{\partial v} = \frac{\partial y_{vvv}}{\partial u}. \tag{11}$$

The second of these yields a relation of the form

$$\alpha y_{uv} + \beta y_u + \gamma y_v + \delta y = 0, \tag{2}$$

which can be satisfied only if $\alpha = \beta = \gamma = \delta = 0$. We obtain, therefore, the equations

$$
\begin{aligned}
a^{(21)}_v + b^{(21)} + a'c^{(21)} &= a^{(12)}_u + ab^{(12)} + c^{(12)},\\
a^{(21)}b^{(12)} + b^{(21)}_v + b'c^{(21)} &= a^{(12)}b^{(21)} + bb^{(12)} + b^{(12)}_u + d^{(12)},\\
a^{(21)}c^{(12)} + c'c^{(21)} + c^{(21)}_v + d^{(21)} &= a^{(12)}c^{(21)} + cb^{(12)} + c^{(12)}_u,\\
a^{(21)}d^{(12)} + d'c^{(21)} + d^{(21)}_v &= a^{(12)}d^{(21)} + db^{(12)} + d^{(12)}_u,
\end{aligned}
\tag{12}
$$

* See E. J. Wilczynski, *Projective differential geometry of curved surfaces* (First Memoir), these Transactions, vol. 8 (1907), p. 239.

which must be identically satisfied. We suppose that they are; it may then be verified without difficulty that the first and third of equations (11) lead to no new relations in the presence of equations (12). All derivatives, of any order, will then be expressible, in one and only one way, as linear combinations of y_{uv}, y_u, y_v, y, in virtue of equations (12). *Equations* (12) *are a necessary and sufficient condition that system* (5) *be completely integrable, in the sense that any fundamental system of solutions of* (5) *be expressible linearly, with constant coefficients, in terms of any other fundamental system.**

It is now apparent why a completely integrable system of form (5) may be made the starting point for our projective theory. Since the configuration defined by means of any fundamental system of solutions is a projective transformation of that defined by any other fundamental system, it follows that geometric properties expressed in terms of the differential equations are common to a net and all of its projective transformations; i.e., they are projective properties.

Certain arbitrary features, however, have been introduced by our analytic formulation. Because of the use of homogeneous coördinates and parametric equations, it is possible to make any proper transformation of the dependent variable of the form

$$y = \lambda(u, v)\,\bar{y}, \tag{13}$$

and any proper transformation of the independent variables of the form

$$\bar{u} = U(u), \qquad \bar{v} = V(v), \tag{14}$$

without disturbing the net defined by equations (1). But these transformations change system (5) into a new system of the same form, in which the coefficients, however, are different. If, then, a combination of the coefficients and variables of system (5) is to have a geometric significance, it must remain essentially unchanged under the transformations (13) and (14). We shall call such a combination of the coefficients of (5), and their derivatives an *invariant*; if the dependent variable y also enter explicitly in the expression, we shall call it a *covariant*.

The calculation of all the invariants and covariants, as thus defined, forms the essence of Wilczynski's method. We shall not perform this calculation in the present paper, because our purpose is to seek the geometric facts, rather than to give a complete formal theory of the system of differential equations.

We deduce from the first of equations (12) a relation which will be very useful subsequently. Substituting for $b^{(21)}$, $c^{(21)}$, $b^{(12)}$, $c^{(12)}$ their values as

* In the above discussion, we have supposed that all of the functions involved are analytic. However, the statement in italics is true even if only the quantities which appear in equations (12) exist. This follows from certain theorems which we have given elsewhere. See G. M. Green, *The linear dependence of functions of several variables and certain completely integrable systems of partial differential equations*, these Transactions, vol. 17 (1916), pp. 483–516.

obtained from (9), we find that

$$a_v^{(21)} + b_v = a_u^{(12)} + c_u'. \tag{15a}$$

This shows that there exists a function f such that

$$f_u = a^{(21)} + b, \qquad f_v = a^{(12)} + c'. \tag{15b}$$

We have supposed that the parametric net is not a conjugate net, and thus were enabled to write the differential equations of the problem in the form (5). We shall find it convenient to write these equations also in the form

$$\begin{aligned} y_{uv} &= \alpha y_{uu} + \beta y_u + \gamma y_v + \delta y, \\ y_{uv} &= \alpha' y_{vv} + \beta' y_u + \gamma' y_v + \delta' y, \end{aligned} \tag{16}$$

where

$$\begin{aligned} &\alpha = \frac{1}{a}, \qquad \beta = -\frac{b}{a}, \qquad \gamma = -\frac{c}{a}, \qquad \delta = -\frac{d}{a}, \\ &\alpha' = \frac{1}{a'}, \qquad \beta' = -\frac{b'}{a'}, \qquad \gamma' = -\frac{c'}{a'}, \qquad \delta' = -\frac{d'}{a'}. \end{aligned} \tag{16a}$$

Of course in doing this we assume that neither a nor a' is zero. Geometrically, this is equivalent to assuming that *neither the curves* C_u ($v = const.$) *nor the curves* C_v ($u = const.$) *are asymptotic.**

2. Certain nets determined by the given one

On the surface S_y determined by equations (1), the net of asymptotic curves is of course fundamental. This net may be found from the given net as follows. By a transformation of the independent variables of the form

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v),$$

the given parametric net is changed into a new parametric net, which is asymptotic if and only if ϕ and ψ satisfy the same quadratic partial differential equation of the first order,†

$$a' \phi_u^2 - 2\phi_u \phi_v + a\phi_v^2 = 0, \qquad a' \psi_u^2 - 2\psi_u \psi_v + a\psi_v^2 = 0. \tag{17}$$

* The study of asymptotic nets belongs more properly to the general theory of curved surfaces, since only one asymptotic net exists on a curved surface. The theory of surfaces has in fact been developed on this basis in the first three of Wilczynski's five memoirs on curved surfaces, these Transactions, vols. 8–10 (1907–1909). Although the case in which only one of the families of curves is composed of asymptotics would appear at first sight to be less interesting on account of its lack of symmetry, it seems nevertheless to be useful in the general theory of congruences, and especially in the theory of W-congruences, the congruence consisting of the tangents to the non-asymptotic curves on the surface.

† E. J. Wilczynski, *Projective differential geometry of curved surfaces*, First Memoir, these Transactions, vol. 8 (1907), pp. 233–260, in particular p. 243.

We may express this otherwise by saying that *the asymptotic net on the surface S_y is determined by the ordinary quadratic differential equation*

$$adu^2 + 2dudv + a'\,dv^2 = 0, \tag{18}$$

whose factors are of course distinct, since $1 - aa' \neq 0$. At any point of the surface, equation (18) determines the two asymptotic tangents. In fact, if $(dv/du)_1$ and $(dv/du)_2$ are the two roots of the quadratic, then

$$y_u + \left(\frac{dv}{du}\right)_1 y_v, \qquad y_u + \left(\frac{dv}{du}\right)_2 y_v$$

are points on the asymptotic tangents.

In general, a quadratic differential equation

$$Adu^2 + 2Bdudv + Cdv^2 = 0, \tag{19}$$

whose discriminant is different from zero, will define a net of curves on the surface, and will determine at any point of the surface the two tangents to the curves of the net passing through that point. The coefficients are of course functions of u and v. If, then, a second net be defined by the differential equation

$$A'\,du^2 + 2B'\,dudv + C'\,dv^2 = 0, \tag{20}$$

the two tangents defined by (19) at any point of the surface will separate harmonically those defined by (20), if and only if the harmonic invariant of the two quadratics vanishes, i.e., if and only if

$$AC' - 2BB' + A'\,C = 0. \tag{21}$$

Again, the pair of tangents defined by (19) and the pair of tangents defined by (20) determine an involution, and the double rays of the involution will be defined by equating to zero the jacobian of the two quadratic forms:

$$(AB' - A'\,B)\,du^2 + (AC' - A'\,C)\,dudv + (BC' - B'\,C)\,dv^2 = 0. \tag{22}$$

The rays defined by equation (22) of course separate harmonically the pair given by (19) and the pair given by (20). We have therefore a means for determining uniquely a third net of curves from any two given ones.

Let us apply this idea to the parametric net and the asymptotic net. We thus determine a net whose tangents at any point separate harmonically the pair of parametric tangents and the pair of asymptotic tangents. But two tangents at a point which separate harmonically the asymptotic tangents at the same point are conjugate, so that the net which we have determined is a conjugate net; we shall call it the *associate conjugate net*. Its differential equation is

$$adu^2 - a'\,dv^2 = 0. \tag{23}$$

We may state our result in this form: *there exists one and only one conjugate net, the associate conjugate net, such that the tangents to the two curves of the net at any point separate harmonically the parametric tangents at that point.*

We have, of course, implicitly supposed that neither family of curves constituting the parametric net is composed of asymptotics, i.e., we have assumed that both a and a' are different from zero, as we did at the end of Section 1.

From the parametric net may be determined still other nets and families of curves. For instance, the family of curves C_u (v = const.) determines a unique conjugate family. The conjugate net of which the curves C_u form a component family is defined by the differential equation

$$adudv + dv^2 = 0, \tag{24}$$

since v = const. is a solution of this equation, and moreover the simultaneous invariant (21) of (18) and (24) is zero. Therefore, *the one-parameter family of curves conjugate to the family C_u (v = const.) is determined by the differential equation*

$$adu + dv = 0. \tag{25}$$

Similarly, *the family conjugate to the family C_v (u = const.) is determined by the differential equation*

$$du + a'\,dv = 0. \tag{26}$$

The conjugate net determined by the family of curves C_u (and also that determined by the curves C_v) will be important later. We have investigated elsewhere* the subject of conjugate nets from the point of view of the present paper, and some of the results of that investigation will be useful. Equation (25) shows that *the transformation $\bar{u} = U(u, v)$, $\bar{v} = v$, where*

$$U_u - aU_v = 0, \tag{27}$$

makes parametric the conjugate net determined by the one-parameter family of curves C_u (v = const.) of the original parametric net.

We shall find it convenient to denote by $\overline{C}_v$ the curves $\bar{u}$ = const. which are conjugate to the curves C_u, and shall then speak of the conjugate net $(C_u, \overline{C}_v)$.

Let us make the transformation just indicated. We have

$$y_u = \bar{y}_u U_u, \qquad y_v = \bar{y}_u U_v + \bar{y}_v, \tag{28}$$

$$\begin{gathered} y_{uu} = \bar{y}_{uu} U_u^2 + \bar{y}_u U_{uu}, \qquad y_{uv} = \bar{y}_{uu} U_u U_v + \bar{y}_{uv} U_u + \bar{y}_u U_{uv}, \\ y_{vv} = \bar{y}_{uu} U_v^2 + 2\bar{y}_{uv} U_v + \bar{y}_{vv} + \bar{y}_u U_{vv}, \end{gathered} \tag{29}$$

* G. M. Green, *Projective differential geometry of one-parameter families of space curves and conjugate nets on a curved surface* (First Memoir), American Journal of Mathematics, vol. 37 (1915), pp. 215–246; (Second Memoir), ibid., vol. 38 (1916), pp. 287–324.

where $\bar{y}_u$, $\bar{y}_{uu}$, etc., denote $\partial y/\partial \bar{u}$, $\partial^2 y/\partial \bar{u}^2$, etc. From the first and second of equations (29), we obtain

$$y_{uu} - a y_{uv} = - a U_u \bar{y}_{uv} + a_u U_v \bar{y}_u ,$$

in which use has been made of equation (27) and of the first of the following two equations which result from differentiation of (27):

$$U_{uu} - a U_{uv} = a_u U_v , \qquad U_{uv} - a U_{vv} = a_v U_v . \tag{30}$$

Using now the first of equations (5), and also equations (28) and (27), and introducing the abbreviations (16a), we find very readily the equation

$$\bar{y}_{uv} = \left(\frac{a_u}{a^2} + \beta + \frac{\gamma}{a} \right) \bar{y}_u + \frac{\gamma}{U_u} \bar{y}_v + \frac{\delta}{U_u} \bar{y} . \tag{31}$$

This is one of the two partial differential equations of the second order which constitute the completely integrable system for the surface referred to the conjugate net $(C_u, \bar{C}_v)$. The other equation may be found as follows. From the second and third of equations (29), we find

$$y_{vv} - a' y_{uv} = \bar{y}_{uu} (U_v^2 - a' U_u U_v) + \bar{y}_{uv} (2 U_v - a' U_u) + \bar{y}_{vv} + \bar{y}_u (U_{vv} - a' U_{uv}) ,$$

which in virtue of (5), (27), (30), and (31) may be reduced to an equation in $\bar{y}_{uu}$, $\bar{y}_{vv}$, $\bar{y}_u$, $\bar{y}_v$, and $\bar{y}$. *If the conjugate net $(C_u, \bar{C}_v)$ be made parametric, the surface S_y is defined by a fundamental system of solutions of the completely integrable system of partial differential equations*

$$\begin{aligned} \bar{y}_{uu} &= \bar{a} \bar{y}_{vv} + \bar{b} \bar{y}_u + \bar{c} \bar{y}_v + \bar{d} \bar{y} , \\ \bar{y}_{uv} &= \bar{b}' \bar{y}_u + \bar{c}' \bar{y}_v + \bar{d}' \bar{y} , \end{aligned} \tag{32}$$

where

$$\begin{aligned} \bar{a} &= \frac{1}{(aa' - 1) U_v^2} , \\ \bar{b} &= \frac{1}{(1 - aa') U_v} \left[ab' + c' + a' a_v - (2 - aa') \left(\frac{a_u}{a^2} + \beta + \frac{\gamma}{a} \right) - (1 - aa') \frac{U_{vv}}{U_v} \right] , \\ \bar{c} &= \frac{1}{(1 - aa') U_v^2} \left(c' - \frac{2 - aa'}{a} \gamma \right) , \\ \bar{d} &= \frac{1}{(1 - aa') U_v^2} \left(d' - \frac{2 - aa'}{a} \delta \right) , \\ \bar{b}' &= \frac{a_u}{a^2} + \beta + \frac{\gamma}{a} , \qquad \bar{c}' = \frac{\gamma}{U_u} , \qquad \bar{d}' = \frac{\delta}{U_u} . \end{aligned} \tag{33}$$

Equations (33) are precisely of the form used in our investigations on conjugate nets, in the papers already cited. In order to make use of the results of these papers, however, it will be necessary to note certain differentiation formulas. From (28) we have, on using (27),

$$\frac{\partial}{\partial \bar{u}} = \frac{1}{U_u}\frac{\partial}{\partial u}, \qquad \frac{\partial}{\partial \bar{v}} = \frac{\partial}{\partial v} - \frac{1}{a}\frac{\partial}{\partial u}, \tag{34}$$

whence

$$\frac{\partial^2}{\partial \bar{u}\partial \bar{v}} = \frac{1}{U_u}\left(\frac{\partial^2}{\partial u \partial v} - \frac{1}{a}\frac{\partial^2}{\partial u^2} + \frac{a_u}{a^2}\frac{\partial}{\partial u}\right),$$

$$\frac{\partial^2}{\partial \bar{v}^2} = \frac{\partial^2}{\partial v^2} - \frac{2}{a}\frac{\partial^2}{\partial u \partial v} + \frac{1}{a^2}\frac{\partial^2}{\partial u^2} + \left(\frac{a_v}{a^2} - \frac{a_u}{a^3}\right)\frac{\partial}{\partial u}. \tag{35}$$

The formulas

$$\frac{\partial U_u}{\partial \bar{v}} = -\frac{a_u}{a}U_v, \qquad \frac{\partial U_v}{\partial \bar{v}} = -\frac{a_v}{a}U_v, \tag{35a}$$

which follow from (34) and (30), will also be found convenient later.

We may now express, in terms of the coefficients and variables of equations (5), some of the important invariants of the conjugate net $(C_u, \bar{C}_v)$. These invariants are given explicitly in terms of the differential equations (32), in our memoirs on conjugate nets. The Laplace-Darboux invariants of the net are*

$$H = \bar{d}' + \bar{b}'\bar{c}' - \bar{b}'_u, \qquad K = \bar{d}' + \bar{b}'\bar{c}' - \bar{c}'_v, \tag{36}$$

so that, writing

$$H = -\frac{1}{aU_u}h, \qquad K = -\frac{1}{aU_u}k, \tag{37}$$

we find in virtue of (33) and (34) that

$$h = d - \gamma\left(2\frac{a_u}{a} + a\beta + \gamma\right) + \gamma_u - a(\alpha_{uu} - \beta_u),$$

$$k = d - \gamma(\gamma + a\beta) - \gamma_u + a\gamma_v. \tag{38}$$

In the first memoir on conjugate nets we denoted by $\mathfrak{B}'$ and $\mathfrak{C}'$ two invariants which, if use be made of equations (24), (29), and (38) of that memoir, are without difficulty seen to have the values

$$\bar{\mathfrak{B}}' = \frac{1}{4}\left(2\bar{b}' + \frac{\bar{c}}{\bar{a}} - \frac{\bar{a}_v}{2\bar{a}}\right), \qquad \bar{\mathfrak{C}}' = \frac{1}{4}\left(2\bar{c}' - \bar{b} + \frac{\bar{a}_u}{2\bar{a}}\right). \tag{39}$$

We have of course written bars over the $\mathfrak{B}'$ and $\mathfrak{C}'$ in accordance with our present notation. We may express the right-hand members of equations (39) in terms of the coefficients and variables of the original system of differential equations (5), by using formulas (33) and (34); writing

* First Memoir, equations (42), (47).

$$\overline{\mathfrak{C}}' = \frac{1}{U_v(1-aa')}\{\overline{\mathfrak{C}}'\}, \tag{40}$$

we find that

$$4\overline{\mathfrak{B}}' = 2\beta + \frac{4-aa'}{a}\gamma - c' + 2\frac{a_u}{a^2} - \frac{a_v}{a} + \frac{1}{2}\frac{\partial}{\partial v}\log(1-aa') - \frac{1}{2a}\frac{\partial}{\partial u}\log(1-aa'),$$

$$4\{\overline{\mathfrak{C}}'\} = (2-aa')\beta + \frac{4-3aa'}{a}\gamma - ab' - c' + (2-aa')\frac{a_u}{a^2} - \frac{a_v}{a} - \frac{1-aa'}{2a}\frac{\partial}{\partial u}\log(1-aa'). \tag{41}$$

The invariants $\overline{\mathfrak{B}}'$ and $\overline{\mathfrak{C}}'$ are, strictly speaking, invariants peculiar to the conjugate net $(C_u, \overline{C}_v)$; for instance, the equation $\overline{\mathfrak{B}}' = 0$ expresses a geometric property of this net* which is independent of the nature of the surface, in the sense that on any surface whatever there exist conjugate nets for which $\overline{\mathfrak{B}}' = 0$. Certain combinations of the invariants $\overline{\mathfrak{B}}'$ and $\overline{\mathfrak{C}}'$, however, express intrinsic properties of the surface itself. For instance, if and only if one of the quantities $\overline{\mathfrak{C}}' + \sqrt{-\bar{a}}\overline{\mathfrak{B}}'$, $\overline{\mathfrak{C}}' - \sqrt{-\bar{a}}\overline{\mathfrak{B}}'$ vanishes, i.e., if and only if $\overline{\mathfrak{C}}'^2 + \bar{a}\overline{\mathfrak{B}}'^2 = 0$, will the surface be a ruled surface; also, the surface will be a quadric if and only if both $\overline{\mathfrak{B}}'$ and $\overline{\mathfrak{C}}'$ are zero.†

In precisely the same way, we may set up the corresponding invariants for the conjugate net $(\overline{C}_u, C_v)$ formed by the family of curves C_v (u = const.) and the family conjugate thereto. The formulas may be written down, however, by changing each of the invariants already found for the conjugate net $(C_u, \overline{C}_v)$ in accordance with the transformation scheme

$$\left\{\begin{matrix} u, & a, & b, & c, & d, & \alpha, & \beta, & \gamma, & \delta \\ v, & a', & c', & b', & d', & \alpha', & \gamma', & \beta', & \delta' \end{matrix}\right\},$$

any letter being replaced by the letter immediately above or below it. Thus, for instance, the Laplace-Darboux invariants—or rather the invariants corresponding to the quantities (38)—are for the conjugate net $(\overline{C}_u, C_v)$

$$h' = d' - \beta'(\beta' + a'\gamma') - \beta'_v + a'\beta'_u,$$

$$k' = d' - \beta'\left(2\frac{a'_v}{a'} + a'\gamma' + \beta'\right) + \beta'_v - a'(\alpha'_{vv} - \gamma'_v). \tag{42}$$

* See page 317 of the second memoir on conjugate nets for the geometric interpretation of the vanishing of either $\overline{\mathfrak{B}}'$ or $\overline{\mathfrak{C}}'$.

† See the first memoir on conjugate nets, p. 239.

3. The congruences of tangents to the families of the net. Laplace transformations

We recall that on the surface S_y, given by the equations

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4), \tag{1 bis}$$

the parametric curves are not asymptotic, and do not form a conjugate net. Let us consider the congruence of tangents to the curves C_u ($v = \text{const.}$). One sheet of its focal surface is the surface S_y; the other sheet may be determined as follows. The point whose coördinates are $y_u^{(1)}, \cdots, y_u^{(4)}$, or, as we may say, the point y_u, is on the tangent to the curve C_u which passes through the point y. Any point on this tangent except the point y is given by the formula

$$\rho = y_u + \lambda y. \tag{43}$$

Consider now the ruled surface whose generators are the tangents to the curves C_u at the points where these curves meet a fixed curve C_v. This ruled surface is not developable, since we have supposed that the curves C_u and C_v are not conjugate. Now, any plane passing through a non-specialized generator of a non-developable ruled surface is tangent to the surface at some point on that generator. The point ρ describes of course a curve on the ruled surface; we shall now determine λ as a function of u and v in such a way that the curve traced by ρ shall be the locus of the points in which the ruled surface is touched by the planes which osculate the curves C_u at the points of the fixed curve C_v. We need only express the condition that the tangent to the curve ρ at any point lies in the plane which osculates at the corresponding point y the corresponding curve C_u. The point ρ is on this tangent; another point on the tangent to the curve traced by ρ as v varies along the fixed curve C_v is given by

$$\rho_v = y_{uv} + \lambda y_v + \lambda_v y,$$

which by the first of equations (16) becomes

$$\rho_v = \alpha y_{uu} + \beta y_u + (\lambda + \gamma) y_v + (\lambda_v + \delta) y.$$

Now, the points y, y_u, y_{uu} determine the osculating plane to the curve C_u at y, therefore the point ρ_v lies in this osculating plane if and only if $\lambda = -\gamma$. This gives us the required expression for ρ, viz.

$$\rho = y_u - \gamma y. \tag{44}$$

We may describe the point ρ geometrically in a slightly different way. Consider two points y and $y + \Delta y$ on a single curve C_v, and the two curves C_u and C_u' which pass through them. The tangent to C_u at y cuts in a point P the plane which osculates C_u' at $y + \Delta y$. As $y + \Delta y$ approaches y along C_v,

the point P approaches the point ρ. If we note that the osculating planes to C_u and C_u' at y and $y + \Delta y$ intersect in a line which passes through P, we may give the following geometric characterization of the quantity ρ. *The tangents to the curves C_u at the points where they meet a fixed curve C_v form a non-developable ruled surface. The osculating planes to the curves C_u at the same points envelop a developable surface. The ruled surface is tangent to the developable surface along a curve which is given by the equations*

$$\rho^{(k)} = y_u^{(k)} - \gamma y^{(k)} \qquad (k = 1, 2, 3, 4)$$

for u = const.

For each point y on the surface S_y, we have a corresponding point ρ, and consequently the totality of points ρ form in general a surface S_ρ. We shall consider the exceptional cases presently. That the surface S_ρ is the second sheet of the focal surface of the congruence of tangents to the curves C_u (v = const.) on S_y follows immediately from the above geometric discussion. In fact, as is well known, the osculating planes to the curves C_u envelop the second focal sheet of the congruence of tangents to these curves. We may also prove this directly. The points ρ, ρ_u, ρ_v determine the tangent plane to the surface S_ρ at ρ. We have

$$\begin{aligned} \rho_u &= y_{uu} - \gamma y_u - \gamma_u y, \\ \rho_v &= \alpha y_{uu} + \beta y_u + (\delta - \gamma_v) y, \end{aligned} \tag{45}$$

so that the said tangent plane is in fact the osculating plane, at the corresponding point y, to the curve v = const. The line joining y and ρ lies in this plane, so that this line is tangent to the surface S_ρ at ρ. *The surface S_ρ is the second focal sheet of the congruence of tangents to the curves C_u (v = const.) on the surface S_y.*

Similarly, *the second focal sheet of the congruence of tangents to the curves C_v (u = const.) on S_y is the surface S_σ defined by the quantity*

$$\sigma = y_v - \beta' y. \tag{46}$$

The analogy of the foregoing to the theory of the Laplace transformations of surfaces referred to conjugate nets, as developed by Darboux, is obvious. To the original surface S_y we have made to correspond, by means of equation (44), the surface S_ρ, and to the net of parameter curves on S_y a net u = const., v = const. on S_ρ. We may regard equation (44) as defining a transformation of S_y into S_ρ. In the same way, equation (46) transforms S_y into S_σ. Now, the same operations may be carried out on the surfaces S_ρ and S_σ, giving rise to two new surfaces for each of these. And so on indefinitely. There is, however, no single continuous sequence as in the Darboux theory, because our original net of curves is not conjugate. Nor can a parametric net on any one of the surfaces derived from S_y by repeated application of the trans-

formations (44) and (46) ever be conjugate, if the surface is non-degenerate and non-developable, because in that case, by Darboux's theory, the parametric net on the surface from which the conjugate net was derived would also be conjugate. It follows that if S_η be any one of the set of non-degenerate, non-developable surfaces derived from S_y by repeated application of the transformations (44) and (46), then the points η_{uv}, η_u, η_v, η must be linearly independent, that is, there must exist between these quantities no relation of the form (2), with non-vanishing coefficients:

$$A\eta_{uv} + B\eta_u + C\eta_v + D\eta = 0.$$

It may happen, however, that such a relation occurs for a surface of the series. That surface must then be developable or degenerate. We investigate this possibility for the surface S_ρ, by obtaining the expressions for ρ, ρ_u, ρ_v, and ρ_{uv}:

$$\begin{aligned}
&\rho = y_u - \gamma y, \qquad \rho_u = a y_{uv} + (b - \gamma) y_u + c y_v + (d - \gamma_u) y, \\
&\rho_v = y_{uv} - \gamma y_v - \gamma_v y, \\
&\rho_{uv} = y_{uuv} - \gamma y_{uv} - \gamma_u y_v - \gamma_v y_u - \gamma_{uv} y \\
&\quad = (a^{(21)} - \gamma) y_{uv} + (b^{(21)} - \gamma_v) y_u \\
&\qquad\qquad + (c^{(21)} - \gamma_u) y_v + (d^{(21)} - \gamma_{uv}) y,
\end{aligned} \tag{47}$$

where $a^{(21)}$, $b^{(21)}$, $c^{(21)}$, $d^{(21)}$ are given by equations (9). We have supposed y, y_u, y_v, y_{uv} linearly independent, so that the determinant of their coefficients in equations (47) must vanish if a relation of form (2) is to exist between ρ, ρ_u, ρ_v, ρ_{uv}; we have, then,

$$\begin{vmatrix} 0 & 1 & 0 & -\gamma \\ a & b-\gamma & c & d-\gamma_u \\ 1 & 0 & -\gamma & -\gamma_v \\ a^{(21)}-\gamma & b^{(21)}-\gamma_v & c^{(21)}-\gamma_u & d^{(21)}-\gamma_{uv} \end{vmatrix} = 0.$$

This condition is very easily found to reduce to

$$[d + \gamma(b - \gamma) - \gamma_u + a\gamma_v](c^{(21)} + \gamma a^{(21)} - \gamma_u - \gamma^2) = 0. \tag{48}$$

Each factor of the left-hand member is of course an invariant; in fact, the first one is the Laplace-Darboux invariant k of the conjugate net $(C_u, \overline{C}_v)$, given by the second of equations (38). We may state then the result, that *the surface S_ρ is developable or degenerate if either of the invariants*

$$\begin{aligned}
k &= d + \gamma(b - \gamma) - \gamma_u + a\gamma_v, \\
G &= c^{(21)} + \gamma a^{(21)} - \gamma_u - \gamma^2
\end{aligned} \tag{49}$$

vanish for the original net on S_y.

Similarly, *the surface S_σ is developable or degenerate if either of the invariants*

$$h' = d' + \beta'(c' - \beta') - \beta'_v + a'\beta'_u,$$
$$G' = b^{(12)} + \beta' a^{(12)} - \beta'_v - \beta'^2 \tag{50}$$

vanish.

The geometric interpretation of the vanishing of G or G' is not difficult. We have by (8)

$$y_{uuu} = a^{(30)} y_{uv} + b^{(30)} y_u + c^{(30)} y_v + d^{(30)} y,$$

which on using the first of equations (16) becomes

$$y_{uuu} = \alpha a^{(30)} y_{uu} + (b^{(30)} + \beta a^{(30)}) y_u + (c^{(30)} + \gamma a^{(30)}) y_v + (d^{(30)} + \delta a^{(30)}) y.$$

*The curves C_u are plane if and only if the quantity $c^{(30)} + \gamma a^{(30)}$ vanishes.** But by using the values of $c^{(30)}$ and $a^{(30)}$ as given by equations (9), we find that

$$c^{(30)} + \gamma a^{(30)} = a(c^{(21)} + \gamma a^{(21)} - \gamma_u - \gamma^2) = aG. \tag{51}$$

The vanishing of the invariant G is a necessary and sufficient condition that the curves v = const. on S_y be plane. The curves u = const. on S_y are plane if and only if the invariant G' vanishes.

If the curves C_u are plane curves, the surface S_ρ is the developable surface enveloped by the planes of these curves; if the curves C_v are plane, the surface S_σ is the developable enveloped by their planes.

The interpretation of the vanishing of the Laplace-Darboux invariant k is well-known. The surface S_ρ in this case degenerates into a curve, and the curves C_u are then conjugate to a family of curves of contact of cones enveloping the surface and having their vertices on a curve in space, viz. the curve into which S_ρ degenerates.†

The cases just considered are the only ones in which the sequence of Laplace transformations for a surface referred to a conjugate net is stopped in either direction. For our surface S_y, however, which is referred to a non-conjugate net, the series of surfaces obtained by successive application of the transformations (44) and (46) may be stopped in other ways. First we note that we have already disposed of the cases in which the quantities ρ_{uv}, ρ_u, ρ_v, ρ are connected by a linear relation. We may therefore suppose that the determinant (3) formed for ρ does not vanish in the case we shall now consider. Then the quantity ρ will satisfy a completely integrable system of form (5). If we eliminate between equations (47) and the equation

* For, if $c^{(30)} + \gamma a^{(30)} = 0$, a linear relation exists between y_{uuu}, y_{uu}, y_u, y; i.e., the determinant $|y^{(k)}, y_u^{(k)}, y_{uu}^{(k)}, y_{uuu}^{(k)}|$ vanishes identically. But for v = const., this determinant is nothing but the wronskian of the four functions $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$. Therefore for a fixed v a linear relation with *constant* coefficients exists between $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$.

† Cf. the first memoir on conjugate nets, p. 236.

$$\begin{aligned}\rho_{uu} &= y_{uuu} - \gamma y_{uu} - 2\gamma_u y_u - \gamma_{uu} y \\ &= (a^{(30)} - a\gamma) y_{uv} + (b^{(30)} - b\gamma - 2\gamma_u) y_u + (c^{(30)} - c\gamma) y_v \\ &\qquad + (d^{(30)} - d\gamma - \gamma_{uu}) y\end{aligned} \tag{52}$$

the quantities y_{uv}, y_u, y_v, y—an elimination which may be made in only one way, since no linear relation connects these quantities—we obtain the first of the following equations:

$$\begin{aligned}\rho_{uu} &= a_{-1}\rho_{uv} + b_{-1}\rho_u + c_{-1}\rho_v + d_{-1}\rho, \\ \rho_{vv} &= a'_{-1}\rho_{uv} + b'_{-1}\rho_u + c'_{-1}\rho_v + d'_{-1}\rho,\end{aligned} \tag{53}$$

where in particular

$$a_{-1} = a. \tag{54}$$

The second of equations (53) is obtained by eliminating y_{uv}, y_u, y_v, y between equations (47) and the equation

$$\begin{aligned}\rho_{vv} &= (a^{(12)} - a'\gamma) y_{uv} + (b^{(12)} - b'\gamma) y_u \\ &\qquad + (c^{(12)} - c'\gamma - 2\gamma_v) y_v + (d^{(12)} - d'\gamma - \gamma_{vv}) y,\end{aligned} \tag{55}$$

and the coefficient a'_{-1} is without much difficulty found to have the value

$$a'_{-1} = \frac{F}{G}, \tag{56}$$

where G is given by (49), and

$$F = c^{(12)} + \gamma a^{(12)} - c'\gamma - a'\gamma^2 - 2\gamma_v. \tag{57}$$

In order, now, that the surface S_ρ be non-developable, we must have $1 - a_{-1}a'_{-1}$ different from zero, i.e., $G - aF$ must not vanish. On using the values for $a^{(12)}$, $c^{(12)}$, $a^{(21)}$, $c^{(21)}$ given by (9), we find that

$$G - aF = k, \tag{58}$$

so that the vanishing of this quantity would lead to nothing new.

There remains, however, the possibility that a'_{-1}, in other words F, may vanish. In this case the curves u = const. on the surface S_ρ are asymptotics, and it is obvious that the congruence of tangents to these curves can have only the one focal sheet S_ρ. We may therefore state the result, that *aside from the cases already considered, in which a surface of the series derived from S_y by means of the transformations* (44) *and* (46) *is developable or degenerate, the only case in which the series is stopped in any direction arises when a parametric family of curves on a surface derived from S_y is asymptotic. The curves v = const. on S_ρ, and u = const. on S_σ, cannot be asymptotic; the curves u = const. on S_ρ are asymptotic if and only if the invariant F given by* (57) *is zero, while neither k nor G vanishes. Similarly, if and only if the invariant*

$$F' = b^{(21)} + \beta' a^{(21)} - b\beta' - a\beta'^2 - 2\beta'_u \tag{59}$$

vanishes, whereas h' and G' are neither of them zero, will the curves $v = const.$ on S_σ be asymptotic.

In terms of invariants just obtained may be expressed two very important ones. A W-congruence is a congruence for which the asymptotic curves on the two focal sheets correspond. We may therefore obtain the condition that the tangents to the curves C_u on S_y constitute a W-congruence by writing down the condition that the differential equation which determines the asymptotic net on S_y is the same as the differential equation which determines the asymptotic net on S_ρ. The asymptotics on S_y are determined by equation (18); the asymptotics on S_ρ are therefore determined by the equation

$$a_{-1}\, du^2 + 2dudv + a'_{-1}\, dv^2 = 0. \tag{60}$$

Recalling that $a_{-1} = a$, we see at once that equations (18) and (60) are identical if and only if $a'_{-1} = a'$. Using equation (56), we find that this is equivalent to the vanishing of the invariant

$$W^{(u)} = F - a' G; \tag{61}$$

and by means of the expressions given by equations (9) for $a^{(12)}$, $c^{(12)}$, $a^{(21)}$, $c^{(21)}$, we find without much difficulty that

$$W^{(u)} = \frac{\partial}{\partial u}(a'\gamma + c') - 2\gamma_v. \tag{62}$$

A necessary and sufficient condition that the congruence of tangents to the curves C_u ($v = const.$) on the surface S_y be a W-congruence, is the vanishing of the invariant $W^{(u)}$ defined by equation (61) *or* (62).

Similarly, *the tangents to the curves C_v ($u = const.$) on S_y form a W-congruence if and only if the invariant*

$$W^{(v)} = F' - aG' = \frac{\partial}{\partial v}(a\beta' + b) - 2\beta'_u \tag{63}$$

vanishes.

4. The axis congruence

In analogy to the moving trihedral of metric differential geometry, it is important to have for our configuration a moving tetrahedron of reference. We have already found what may serve as three vertices of the tetrahedron, viz. the points y, ρ, σ, and the two edges $y\rho$ and $y\sigma$. We now determine the third edge through y, and on it the fourth vertex of the tetrahedron, as follows.

Since the two curves C_u and C_v passing through a point y are not asymptotic, their osculating planes at the point y intersect in a line passing through y

and not lying in the tangent plane to the surface at y. This line we shall take as a third edge of our tetrahedron; we shall call it the *axis** of the point y. It remains, therefore, to find thereon a covariant point to serve as the fourth vertex.

We first define the axis analytically. If we equate the two values of y_{uv} given by equations (16), and then make suitable transpositions, we find that

$$\alpha y_{uu} + (\beta - \beta')\, y_u + \delta y = \alpha'\, y_{vv} + (\gamma' - \gamma)\, y_v + \delta'\, y\,.$$

The left-hand member represents a point in the osculating plane to the curve C_u at y, the right-hand member a point in the osculating plane to the curve C_v at y. Consequently either of these expressions represents some point z on the line of intersection of the two osculating planes. Using equations (5) and (16a), we may find an unambiguous expression for this point: *the axis of the point y on the surface S_y is the line joining the point y with the point z determined by the expression*

$$z = y_{uv} - \beta'\, y_u - \gamma y_v\,. \tag{64}$$

To each point y of the surface corresponds a definite axis yz. The totality of axes form a congruence, which we shall call the *axis congruence*. We may then determine the two sheets of the focal surface of the axis congruence, and thereby obtain on any line yz of the congruence two covariant points, viz., the focal points of the line yz. We shall find presently that the analytic expressions for these focal points are rather complicated, and in fact involve irrationalities; we obtain, however, a unique covariant point by taking the harmonic conjugate of y with respect to the two focal points. That this choice is a very natural one, and one which leads to some important consequences, will appear in the sequel.

Any point on the line yz, with the exception of the point y, is given by the expression

$$\eta = z + \lambda y = y_{uv} - \beta'\, y_u - \gamma y_v + \lambda y\,.$$

We wish first to determine λ as a function of u and v in such a way that the surface S_η thus defined shall be a focal sheet of the axis congruence. There will therefore in general be two such functions $\lambda(u, v)$. The tangent plane to the surface S_η at any point η is determined by the three points η, η_u, η_v, so that if S_η is to be a focal sheet of the axis congruence this plane must contain in it the line yz, or in other words the point y. Therefore we must obtain the condition that the four quantities y, η, η_u, η_v be linearly dependent, or what is the same thing, that y, z, η_u, η_v be linearly dependent. We have

* This name was introduced by Wilczynski for the corresponding case in which the parameter net on S_y is conjugate. See his paper *The general theory of congruences*, these Transactions, vol. 16 (1915), p. 314.

$$
\begin{aligned}
z_u &= y_{uuv} - \beta' y_{uu} - \gamma y_{uv} - \beta'_u y_u - \gamma_u y_v \\
&= (a^{(21)} - a\beta' - \gamma) y_{uv} + (b^{(21)} - b\beta' - \beta'_u) y_u \\
&\qquad + (c^{(21)} - c\beta' - \gamma_u) y_v + (d^{(21)} - d\beta') y, \qquad (65) \\
z_v &= (a^{(12)} - a'\gamma - \beta') y_{uv} + (b^{(12)} - b'\gamma - \beta'_v) y_u \\
&\qquad + (c^{(12)} - c'\gamma - \gamma_v) y_v + (d^{(12)} - d'\gamma) y,
\end{aligned}
$$

so that

$$
\begin{aligned}
\eta_u &= (a^{(21)} - a\beta' - \gamma) y_{uv} + (b^{(21)} - b\beta' - \beta'_u + \lambda) y_u \\
&\qquad + (c^{(21)} - c\beta' - \gamma_u) y_v + (d^{(21)} - d\beta' + \lambda_u) y, \qquad (65a) \\
\eta_v &= (a^{(12)} - a'\gamma - \beta') y_{uv} + (b^{(12)} - b'\gamma - \beta'_v) y_u \\
&\qquad + (c^{(12)} - c'\gamma - \gamma_v + \lambda) y_v + (d^{(12)} - d'\gamma + \lambda_v) y.
\end{aligned}
$$

Also,

$$
z = y_{uv} - \beta' y_u - \gamma y_v .
$$

Now, the points y, z, η_u, η_v will lie in a plane if and only if the determinant of the coefficients of y_{uv}, y_u, and y_v in the above expressions for η_u, η_v, and z is zero. This determinant when expanded leads to a quadratic equation in λ:

$$
\begin{aligned}
&\lambda^2 + (F + F' + \beta'_u + \gamma_v - 2\beta'\gamma)\lambda \\
&\qquad + (F + \gamma_v - \beta'\gamma)(F' + \beta'_u - \beta'\gamma) - GG' = 0, \qquad (66)
\end{aligned}
$$

where F, F', G, and G' are given by (49), (50), (57), and (59). *The focal sheets of the axis congruence are given by the expressions*

$$
\tau_1 = z + \lambda_1 y, \qquad \tau_2 = z + \lambda_2 y, \qquad (67)
$$

where λ_1 *and* λ_2 *are the roots of the quadratic* (66).

The actual explicit expressions for λ_1 and λ_2 are of course complicated. We may, however, obtain a covariant point with a neater expression, and which is moreover uniquely determined, by choosing the point τ which is the harmonic conjugate of the point y with respect to the corresponding focal points τ_1, τ_2. We find without difficulty that

$$
\begin{aligned}
\tau &= z + \tfrac{1}{2}(\lambda_1 + \lambda_2) y \\
&= z - \tfrac{1}{2}(F + F' - 2\beta'\gamma + \beta'_u + \gamma_v) y. \qquad (68)
\end{aligned}
$$

The discriminant of the quadratic (66) is easily found to be

$$
\Delta = (F - F' + \gamma_v - \beta'_u)^2 + 4GG'. \qquad (69)
$$

Its vanishing is of course a necessary and sufficient condition that the focal

sheets of the axis congruence coincide, in other words that the axis congruence consists of the tangents to one of the two families of asymptotics on the focal surface. It is obvious that on any curved surface there exist nets for which the axis congruence has this property; it will also follow from our subsequent discussion that a given congruence formed of the tangents to a family of asymptotics on a surface is the axis congruence of an indefinite number of nets of curves on any surface cutting the lines of the congruence.

5. The ruled surfaces of the axis congruence

The lines yz of the axis congruence which pass through the points of a fixed curve C_u (v = const.) on the surface S_y generate a ruled surface $R^{(u)}$. Similarly, to every curve C_v (u = const.) on S_y corresponds a ruled surface $R^{(v)}$ of the axis congruence. We shall call the surfaces $R^{(u)}$ and $R^{(v)}$ the *parametric ruled surfaces* of the axis congruence.

The osculating plane to a curve C_u at any point y contains in it the axis of that point, so that the said osculating plane is tangent to the ruled surface $R^{(u)}$ at y. Therefore, *the surface S_y intersects the parametric ruled surfaces in asymptotic curves on the latter.*

If a parametric ruled surface $R^{(u)}$ be a developable, then the osculating plane to the curve C_u at any point is tangent to $R^{(u)}$ along the entire generator through that point, so that the developable is the envelope of the osculating planes of C_u. But the axes yz corresponding to the points of C_u can not be tangents of the curve C_u, as they would have to be if C_u were a twisted curve. Consequently, *the ruled surfaces $R^{(u)}$ will be developable if and only if the corresponding curve C_u is a plane curve, and the plane of the curve will be the developable.*

Let us suppose that the ruled surfaces $R^{(u)}$ are not developable, and let us fix our attention on a single point y of the surface S_y. Through this point passes a single curve C_v, and also a single parametric ruled surface $R^{(u)}$. Then the plane which osculates the curve C_v at y passes through the generator yz and is therefore tangent to the ruled surface $R^{(u)}$ at some point of yz. We proceed to find this point of tangency. Any point of the line yz, except the point y, is given by an expression of the form

$$\eta = z + \lambda y.$$

As the point y moves along the curve C_u, the line generates the ruled surface $R^{(u)}$, and the point η traces a curve on this ruled surface. We wish to determine λ in such a way that the tangent to the curve described by η shall lie in the plane which osculates the curve C_v at y. Using the first of equations (65a), and the second of equations (16), we find without difficulty that

$$
\begin{aligned}
\eta_u = {} & \alpha'(a^{(21)} - a\beta' - \gamma)\, y_{vv} \\
& + [\, b^{(21)} - b\beta' - \beta'_u + \beta'(a^{(21)} - a\beta' - \gamma) + \lambda\,]\, y_u \\
& + [\, c^{(21)} - c\beta' - \gamma_u + \gamma'(a^{(21)} - a\beta' - \gamma)\,]\, y_v \\
& + [\, d^{(21)} - d\beta' + \delta'(a^{(21)} - a\beta' - \gamma) + \lambda_u\,]\, y\,.
\end{aligned} \tag{70}
$$

Now η_u is a point on the tangent to the curve traced out by η, while the osculating plane to the curve C_v at y is determined by the points y, y_v, y_{vv}. Therefore the point η_u lies in this plane if and only if the coefficient of y_u in the expression just found for η_u is zero. This condition determines λ, which according to (59) has then the value $-F' + \beta'\gamma - \beta'_u$. *The osculating plane to a curve C_v ($u = const.$) at a point y of the surface S_y is tangent at the point*

$$\eta = z - (F' - \beta'\gamma + \beta'_u)\, y \tag{71}$$

to the parametric ruled surface $R^{(u)}$ which passes through the point y.

Similarly, *the osculating plane to the curve C_u at the point y is tangent to the ruled surface $R^{(v)}$ at the point*

$$\eta' = z - (F - \beta'\gamma + \gamma_v)\, y\,. \tag{72}$$

The harmonic conjugate of the point y with respect to the two covariant points η and η' is precisely the point τ given by equation (68). We recall that τ was first defined as the harmonic conjugate of the point y with respect to the focal points on the line yz of the axis congruence. It is therefore natural to inquire under what conditions the points η, η' coincide with the focal points. Of course if one of them is a focal point the other must be also.

Consider the surface S_η formed by all the points η corresponding to the points of the surface S_y. The surface S_η is a focal sheet of the axis congruence if and only if the tangent plane at a point η contains in it the line yz passing through that point. In other words, we seek the condition that the points y, z, η_u, η_v lie in a plane. If in equations (65a) we put for λ its value, $-(F' - \beta'\gamma + \beta'_u)$, and recall that

$$z = y_{uv} - \beta' y_u - \gamma y_v\,,$$

we obtain the following expressions for η_u and η_v:

$$
\begin{aligned}
\eta_u &= (a^{(21)} - a\beta' - \gamma)\, z + G y_v + (\ )\, y\,, \\
\eta_v &= (a^{(12)} - a'\gamma - \beta')\, z + G' y_u + (\ )\, y_v + (\ )\, y\,,
\end{aligned} \tag{73}
$$

where the coefficient of y in the first equation, and the coefficients of y_v and y in the second equation, do not concern us. The quantities G and G' are the

sheets of the axis congruence coincide, in other words that the axis congruence consists of the tangents to one of the two families of asymptotics on the focal surface. It is obvious that on any curved surface there exist nets for which the axis congruence has this property; it will also follow from our subsequent discussion that a given congruence formed of the tangents to a family of asymptotics on a surface is the axis congruence of an indefinite number of nets of curves on any surface cutting the lines of the congruence.

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If a parametric ruled surface $R^{(u)}$ be a developable, then the osculating plane to the curve C_u at any point is tangent to $R^{(u)}$ along the entire generator through that point, so that the developable is the envelope of the osculating planes of C_u. But the axes yz corresponding to the points of C_u can not be tangents of the curve C_u, as they would have to be if C_u were a twisted curve. Consequently, *the ruled surfaces $R^{(u)}$ will be developable if and only if the corresponding curve C_u is a plane curve, and the plane of the curve will be the developable.*

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$$
\begin{aligned}
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& + [\, b^{(21)} - b\beta' - \beta'_u + \beta'(a^{(21)} - a\beta' - \gamma) + \lambda\,]\, y_u \\
& + [\, c^{(21)} - c\beta' - \gamma_u + \gamma'(a^{(21)} - a\beta' - \gamma)\,]\, y_v \\
& + [\, d^{(21)} - d\beta' + \delta'(a^{(21)} - a\beta' - \gamma) + \lambda_u\,]\, y\,.
\end{aligned} \tag{70}
$$

Now η_u is a point on the tangent to the curve traced out by η, while the osculating plane to the curve C_v at y is determined by the points y, y_v, y_{vv}. Therefore the point η_u lies in this plane if and only if the coefficient of y_u in the expression just found for η_u is zero. This condition determines λ, which according to (59) has then the value $-F' + \beta'\gamma - \beta'_u$. *The osculating plane to a curve C_v ($u = const.$) at a point y of the surface S_y is tangent at the point*

$$\eta = z - (F' - \beta'\gamma + \beta'_u)\, y \tag{71}$$

to the parametric ruled surface $R^{(u)}$ which passes through the point y.

Similarly, *the osculating plane to the curve C_u at the point y is tangent to the ruled surface $R^{(v)}$ at the point*

$$\eta' = z - (F - \beta'\gamma + \gamma_v)\, y\,. \tag{72}$$

The harmonic conjugate of the point y with respect to the two covariant points η and η' is precisely the point τ given by equation (68). We recall that τ was first defined as the harmonic conjugate of the point y with respect to the focal points on the line yz of the axis congruence. It is therefore natural to inquire under what conditions the points η, η' coincide with the focal points. Of course if one of them is a focal point the other must be also.

Consider the surface S_η formed by all the points η corresponding to the points of the surface S_y. The surface S_η is a focal sheet of the axis congruence if and only if the tangent plane at a point η contains in it the line yz passing through that point. In other words, we seek the condition that the points y, z, η_u, η_v lie in a plane. If in equations (65a) we put for λ its value, $-(F' - \beta'\gamma + \beta'_u)$, and recall that

$$z = y_{uv} - \beta' y_u - \gamma y_v\,,$$

we obtain the following expressions for η_u and η_v:

$$
\begin{aligned}
\eta_u &= (a^{(21)} - a\beta' - \gamma)z + Gy_v + (\ )\, y\,, \\
\eta_v &= (a^{(12)} - a'\gamma - \beta')z + G'\, y_u + (\ )\, y_v + (\ )\, y\,,
\end{aligned} \tag{73}
$$

where the coefficient of y in the first equation, and the coefficients of y_v and y in the second equation, do not concern us. The quantities G and G' are the

invariants given by (49) and (50), whose vanishing is a necessary and sufficient condition that the curves C_u and C_v be respectively plane curves.

Equations (73) show that the points y, z, η_u, η_v lie in the same plane if and only if $GG' = 0$. We have therefore the result: *the surfaces S_η and $S_{\eta'}$ are the focal sheets of the axis congruence if and only if one of the families of curves of the parametric net C_u, C_v on the surface S_y consist of plane curves.*

We see, then, that the points η, η' do not in general coincide with the focal points, and in recapitulation we may describe our configuration as follows. Suppose that neither of the component families C_u, C_v of the given net of curves on the surface S_y consists of plane curves or asymptotics.* At any point y of the surface, the osculating planes to the curves C_u, C_v which pass through that point intersect in a line, the axis of the point y, which is not tangent to the surface at y. The totality of these lines form a congruence, the axis congruence, the two focal sheets of which are touched by each line of the congruence in two focal points on that line. The harmonic conjugate of the point y with respect to the two focal points gives a point τ covariantly associated with y. Consider now the curves C_u and C_v which pass through the point y. The lines of the axis congruence which cut the curve C_u form a non-developable ruled surface $R^{(u)}$, and those which cut the curve C_v form a non-developable ruled surface $R^{(v)}$. The osculating plane to the curve C_v at y is tangent to the ruled surface $R^{(u)}$ at a point η on the axis of y, and the osculating plane to the curve C_u at y is tangent to the ruled surface $R^{(v)}$ at a point η' on the same axis. The harmonic conjugate of the point y with respect to the points η, η' is the point τ, which is also the harmonic conjugate of y with respect to the focal points of the axis. That is, the points η, η' are harmonically separated by the points y, τ, and these in turn by the focal points.

The above discussion suggests at once some new investigations in the general theory of congruences. First of all, given a rectilinear congruence Γ, and a surface S_y such that one and only one line yz of the congruence passes through each point y of the surface, can a net of curves be determined on S_y for which the congruence Γ is the axis congruence? The question is identical with a familiar one in metric geometry, viz., the determination of the geodesics on a surface. Recalling that a geodesic is characterized by the property that the osculating plane to the curve at any point contains in it the normal to the surface at that point, we see that if the given congruence Γ is the congruence of normals to the surface, this congruence is the axis congruence for any net of curves on the surface formed of geodesics. Our general

* In our analytic work, we have found it convenient to suppose that the net is not conjugate. Our geometric description will, however, be applicable to a conjugate net, as we have in fact shown in the second memoir on conjugate nets already cited.

problem, then, is the determination of all the curves on the surface S_y which have the property that the osculating plane at any point of one of the curves contains in it that line yz of the congruence Γ which passes through the point. It may be shown, exactly as in the case of geodesics, that there exist on the surface S_y a two-parameter family of curves, or as we may say ∞^2 curves, which have the desired property.* Any net formed of two one-parameter families of such curves will then have the congruence Γ as its axis congruence.

Let us select a net of curves of the kind just described. We may apply the geometric results obtained above to this net and its axis congruence, Γ, and thus fix, on the line of the congruence passing through y, the two points η, η'. By choosing a different net from among our ∞^2 curves on S_y, or by considering an entirely different surface through y—referred of course to a net of curves bearing the above characteristic relation to the congruence Γ—we obtain a new pair of points η, η' on the line through y. In this way, keeping y fixed, we obtain on the line an infinite number of pairs of points η, η', which, as we know, are pairs of an involution. The double points of the involution are the point y and the associated point τ. Similarly, for any other point y of the same line of the congruence may be obtained a related point τ. The associated points y and τ are in fact pairs of an involution, the double points of which are the focal points of the congruence.

Of course, the above description will lend itself to refinement, but since our discussion in the present paper is centered about a net of curves, we shall not examine more closely these questions in the general theory of congruences.

The considerations of the present section are readily exemplified in metric geometry. As we have already pointed out, if the parametric net of curves on the surface is composed of geodesics, the axis congruence is the congruence of normals to the surface. The focal sheets of this congruence are the surfaces of centers, or, in other words, the focal points are the principal centers of curvature. It is hardly necessary to state the theorems which result here from our general discussion. One of them, however, deserves explicit mention. If R_1 and R_2 denote the principal radii of curvature at a point y of the

* We have suggested here a new class of curves on a surface, which form a very natural projective generalization of geodesics. Given a surface, we may replace the congruence of its normals by another congruence also uniquely determined by the surface, but in a purely projective way. We refer to the directrix congruence of the second kind—a congruence introduced by Wilczynski in his second memoir on curved surfaces, these Transactions, vol. 9 (1908), pp. 79–120. The ∞^2 geodesics will then be replaced by the ∞^2 curves which are characterized by the property that the osculating plane to any one of the curves at any point contains the line of the directrix congruence of the second kind passing through that point.

The present writer has learned that Miss Pauline Sperry has investigated independently this very generalization of geodesics, in her Chicago dissertation, published in the American Journal of Mathematics, vol. 40 (1918), pp. 213–224.

surface, then the focal points τ_1 and τ_2 of the normal to the surface at y are at distances R_1 and R_2 from y. The point τ, which is the harmonic conjugate of y with respect to τ_1 and τ_2, is easily seen to be at a distance R from y given by

$$R = \frac{2R_1 R_2}{R_1 + R_2} = \frac{2}{\frac{1}{R_1} + \frac{1}{R_2}}.$$

Now, if the distance R is the same for each point y of the surface S_y, the surface S_τ generated by the points τ is parallel to the surface S_y. In that case the mean curvature $1/R_1 + 1/R_2$ is also constant. Conversely, if the mean curvature is constant, the surface S_τ is parallel to the surface S_y. We may therefore state the result, which gives a geometric characterization of surfaces of constant mean curvature:

On the normal to a surface S_y at each point y, mark the principal centers of curvature, and let the point τ be the harmonic conjugate of y with respect to these two points. Then the suface S_y is a surface of constant mean curvature if and only if the surface S_τ formed by the points τ is parallel to the surface S_y. In that case the surface S_τ will also be of constant mean curvature.

6. The developables of the axis congruence

To every curve on the surface S_y corresponds a ruled surface of the axis congruence. We proceed now to determine the curves on S_y which correspond to the developables of this congruence. According to the general theory of congruences, there will in general be two one-parameter families of such developables, and consequently a net of curves on S_y corresponding thereto.

As the point y traces out a curve on the surface S_y, the corresponding axis yz generates a ruled surface. The point $dy = y_u\,du + y_v\,dv$ lies on the tangent to the curve traced by the point y, and the point $dz = z_u\,du + z_v\,dv$ lies on the tangent to the curve traced by the corresponding point z. In order that the said ruled surface generated by the line yz be a developable, it is necessary and sufficient that the four points y, z, dy, dz lie in a plane. From equations (65), we find that

$$\begin{aligned} dz &= z_u\,du + z_v\,dv \\ &= [(a^{(21)} - a\beta' - \gamma)\,du + (a^{(12)} - a'\gamma - \beta')\,dv]\,y_{uv} \\ &\quad + [(b^{(21)} - b\beta' - \beta_u)\,du + (b^{(12)} - b'\gamma - \beta'_v)\,dv]\,y_u \\ &\quad + [(c^{(21)} - c\beta' - \gamma_u)\,du + (c^{(12)} - c'\gamma - \gamma_v)\,dv]\,y_v \\ &\quad + [(d^{(21)} - d\beta')\,du + (d^{(12)} - d'\gamma)\,dv]\,y. \end{aligned}$$

We also have

$$dy = y_u\,du + y_v\,dv, \qquad z = y_{uv} - \beta' y_u - \gamma y_v .$$

If, then, the points y, z, dy, dz are to lie in a plane, the determinant of the coefficients of y_{uv}, y_u, y_v which appear in the expressions for z, dy, and dz must vanish. The vanishing of this determinant is moreover sufficient as well as necessary. On expanding the said determinant and equating to zero, a quadratic differential equation is obtained which, if use be made of equations (49), (50), (57), and (59), is without difficulty found to be

$$G du^2 + (F - F' + \gamma_v - \beta_u')\,du dv - G'\,dv^2 = 0 . \tag{74}$$

From the general discussion given at the beginning of Section 2, it follows that the differential equation (74) defines a net of curves on the surface S_y. We shall call this net the *axis net*, and the curves of the net the *axis curves*. Through a point y of the surface pass in general two curves of the net; we shall call the tangents to these curves at the point y the *axis tangents* of the point y.

The quadratic (74) will of course define only a one-parameter family of curves if the discriminant

$$\Delta = (F - F' + \gamma_v - \beta_u')^2 + 4GG'$$

vanishes. This discriminant is the same as the discriminant of the quadratic (66) which determined the focal sheets of the axis congruence, as was to be expected.

From equation (74) we may read off at once the result which we have already proved geometrically, that the parametric ruled surfaces—i.e., the ruled surfaces of the axis congruence which correspond to the parametric curves C_u and C_v on the surface S_y—are developable if and only if G and G' are zero, in other words, if and only if the curves C_u and C_v are plane curves.

We see also that *the axis tangents corresponding to a point y separate harmonically the parametric tangents at y if and only if*

$$F - F' + \gamma_v - \beta_u' = 0 . \tag{75}$$

If we refer to equations (71) and (72), which define the two covariant points η and η', we find that equation (75) is a necessary and sufficient condition that these two points coincide. We may therefore state the following geometric criterion: *the axis tangents at a point y of the surface separate harmonically the parametric tangents if and only if the two points coincide in which the osculating planes to the curves C_u and C_v at y are tangent to the parametric ruled surfaces $R^{(v)}$ and $R^{(u)}$ respectively.*

Regarding equation (74) as a binary quadratic, and recalling that the asymptotic curves of the surface are given by the quadratic

(18 bis) $$adu^2 + 2dudv + a'dv^2 = 0,$$

we may find the condition that the axis curves form a conjugate net, i.e., that the axis tangents separate harmonically the asymptotic tangents. This condition will be the vanishing of the simultaneous invariant (21), which gives

$$a'G - aG' - F + F' - \gamma_v + \beta'_u = 0.$$

Using equations (61) and (63), we may state the result in the following form: *the axis curves form a conjugate net if and only if*

(76) $$W^{(v)} - W^{(u)} - \gamma_v + \beta'_u = 0.$$

Let $(dv/du)_1$ and $(dv/du)_2$ denote the roots of equation (74) considered as a quadratic in dv/du. Then the axis tangents of the point y are the lines joining y to the points

$$y_u + \left(\frac{dv}{du}\right)_1 y_v, \qquad y_u + \left(\frac{dv}{du}\right)_2 y_v.$$

Using this fact, we may show without difficulty that *the plane determined by the axis and either of the axis tangents is tangent to each of the parametric ruled surfaces $R^{(u)}$ and $R^{(v)}$ at one of the two focal points.*

This last proposition may also be proved geometrically, if it be remembered that a plane which is tangent to a developable of a congruence is also tangent to one of the focal sheets.

7. The ray congruence and its developables

In Section 3 were introduced the covariant points

$$\rho = y_u - \gamma y, \qquad \sigma = y_v - \beta' y,$$

which we may call respectively the minus first and first Laplace transforms of the point y with respect to the parametric net (C_u, C_v). Let us call the line joining the points ρ and σ the *ray** of the point y; it is a line which lies in the tangent plane to the surface. The rays corresponding to all the points of the surface form a congruence, which we shall call the *ray congruence.**

In general, the ray congruence will have two focal sheets, and on each ray there will be two focal points, i.e., the points at which the ray is tangent to the focal sheets. We may determine the focal points on a ray $\rho\sigma$ as follows. Any point on the ray is given by an expression of the form

(77) $$R = \rho + \mu\sigma.$$

* The names *ray* and *ray congruence* were applied by Wilczynski to the corresponding case in which the parametric net is conjugate. See his memoir, *The general theory of congruences*, these Transactions, vol. 16 (1915), pp. 311–327.

The totality of points R corresponding to the points y of the surfaces S_y form a surface S_R. We wish to determine μ as a function of u, v so that the surface S_R shall be a focal sheet, i.e., so that the line $\rho\sigma$ shall be tangent to the surface S_R. In other words, the points ρ, σ, R_u, R_v are to lie in a plane. We have

$$\begin{aligned}
\rho_u &= a y_{uv} + (b - \gamma) y_u + c y_v + (d - \gamma_u) y, \\
\rho_v &= y_{uv} - \gamma y_v - \gamma_v y, \\
\sigma_u &= y_{uv} - \beta' y_u - \beta'_u y, \\
\sigma_v &= a' y_{uv} + b' y_u + (c' - \beta') y_v + (d' - \beta'_v) y,
\end{aligned} \tag{78}$$

so that

$$\begin{aligned}
R_u - \mu_u \sigma &= \rho_u + \mu \sigma_u \\
&= (a + \mu) y_{uv} + (b - \gamma - \beta' \mu) y_u + c y_v + (d - \gamma_u - \mu \beta'_u) y, \\
R_v - \mu_v \sigma &= \rho_v + \mu \sigma_v \\
&= (1 + a' \mu) y_{uv} + b' \mu y_u + [(c' - \beta') \mu - \gamma] y_v \\
&\qquad + [(d' - \beta'_v) \mu - \gamma_v] y.
\end{aligned}$$

Also,

$$\rho = y_u - \gamma y, \qquad \sigma = y_v - \beta' y.$$

If, now, the points ρ, σ, R_u, R_v are to lie in a plane, it is necessary and sufficient that in the expressions just written for ρ, σ, $R_u - \mu_u \sigma$, $R_v - \mu_v \sigma$ the determinant of the coefficients of y_{uv}, y_u, y_v, y vanish. This yields a quadratic equation for μ, which is at once found to be

$$h' \mu^2 + [a h' - a' k + (1 - a a')(\beta'_u - \gamma_v)] \mu - k = 0, \tag{79}$$

where h' and k are invariants given by equations (49) and (50). *The focal sheets of the ray congruence are given by the covariants*

$$R_1 = \rho + \mu_1 \sigma, \qquad R_2 = \rho + \mu_2 \sigma, \tag{80}$$

where μ_1 and μ_2 are the two roots of the quadratic (79).

We may write the quadratic (79) in two other forms which are sometimes more useful. From (58) and (61) we find that

$$k = (1 - a a') G - a W^{(u)}, \qquad a' k = (1 - a a') F - W^{(u)}. \tag{81}$$

Similarly, we have the relations

$$h' = (1 - a a') G' - a' W^{(v)}, \qquad a h' = (1 - a a') F' - W^{(v)}. \tag{82}$$

Equation (79) may therefore be written in either of the following ways:

$$h' \mu^2 + [W^{(u)} - W^{(v)} + (1 - a a')(F' - F + \beta'_u - \gamma_v)] \mu - k = 0, \tag{83}$$

$$\left(G' - \frac{a' W^{(v)}}{1 - aa'}\right)\mu^2 + \left[F' - F + \beta_u' - \gamma_v + \frac{W^{(u)} - W^{(v)}}{1 - aa'}\right]\mu - \left(G - \frac{aW^{(u)}}{1 - aa'}\right) = 0. \tag{84}$$

We recall that the axis curves, i.e., the curves in which the developables of the axis congruence cut the surface S_y, are defined by the differential equation

$$Gdu^2 + (F - F' + \gamma_v - \beta_u')\,dudv - G'\,dv^2 = 0. \tag{74 bis}$$

Consequently the axis tangents of a point y of the surface are the lines joining that point with the points

$$\rho + \left(\frac{dv}{du}\right)_1 \sigma, \qquad \rho + \left(\frac{dv}{du}\right)_2 \sigma,$$

where $(dv/du)_1$ and $(dv/du)_2$ are the roots of the quadratic (74). Using (84) as the equation for the determination of the μ_1 and μ_2 of (80), we see that the points R_1 and R_2, the focal points of the ray, lie on the axis tangents if and only if $W^{(u)}$ and $W^{(v)}$ are both zero. *The axis tangents at each point of the surface S_y intersect the corresponding ray in the focal points of the ray if and only if both of the congruences of tangents to the parametric curves C_u and C_v on S_y are W-congruences.*

We proceed now to set up the differential equation which defines the developables of the ray congruence. As the point y traces out a curve on the surface S_y, the corresponding rays $\rho\sigma$ generate a ruled surface of the ray congruence. We seek those curves on S_y for which the corresponding ruled surfaces of the ray congruence are developable. We shall call these curves the *ray curves* of the surface. They are of course determined by the particular parametric net to which the surface is referred.

If the line $\rho\sigma$ is to generate a developable, the four points ρ, σ, $d\rho = \rho_u\,du + \rho_v\,dv$, $d\sigma = \sigma_u\,du + \sigma_v\,dv$ must lie in a plane. Using equations (78), we find immediately that

$$d\rho = (adu + dv)\,y_{uv} + (b - \gamma)\,du \cdot y_u + (cdu - \gamma dv)\,y_v + [(d - \gamma_u)\,du - \gamma_v\,dv]\,y,$$

$$d\sigma = (du + a'\,dv)\,y_{uv} + (-\beta'\,du + b'\,dv)\,y_u + (c' - \beta')\,dv \cdot y_v + [-\beta_u'\,du + (d' - \beta_v')\,dv]\,y.$$

Equating to zero the determinant formed from the coefficients of y_{uv}, y_u, y_v, y in the expressions for $d\rho$, $d\sigma$, and $\rho = y_u - \gamma y$, $\sigma = y_v - \beta' y$, we obtain the condition that these last four points lie in a plane. On expanding the determinant, it is found without difficulty that *the ray curves of the surface S_y are defined by the differential equation*

$$[k + a(\beta_u' - \gamma_v)]du^2 + [a'k - ah' + (1 + aa')(\beta_u' - \gamma_v)]dudv - [h' - a'(\beta_u' - \gamma_v)]dv^2 = 0, \tag{85}$$

where the Laplace-Darboux invariants k and h' are given by equations (49) and (50).

As was to be expected, the ray curves form in general a net of curves on the surface S_y. We shall call the two tangents at a point y to the two curves of the net which pass through y, the *ray tangents* of the point y.

A comparison of equations (79) and (85) will show at once that *the ray tangents meet the ray in the focal points of the ray if and only if*

$$\beta_u' - \gamma_v = 0. \tag{86}$$

We recall once more that the asymptotic curves of the surface S_y are defined by the quadratic (18),

$$adu^2 + 2dudv + a'dv^2 = 0.$$

The harmonic invariant of this and (85) is $(1 - aa')(\beta_u' - \gamma_v)$. This can vanish if and only if $\beta_u' - \gamma_v = 0$, since we have supposed $1 - aa'$ to be nowhere zero. Combining this with the result previously obtained, we see that *the ray curves form a conjugate net if and only if* $\beta_u' - \gamma_v = 0$, *that is, if and only if at each point of the surface the ray tangents meet the corresponding ray in the focal points of the ray.*

An interesting special case of this last theorem is that in which the axis curves also form a conjugate net. In that case equation (76) is satisfied in addition to equation (86). *The ray curves and axis curves both form conjugate nets if and only if*

$$\beta_u' - \gamma_v = 0, \qquad W^{(u)} - W^{(v)} = 0. \tag{87}$$

In virtue of equations (81) and (82), the differential equation of the axis curves, (74), may be written

$$(k + aW^{(u)})du^2 + [a'k - ah' + W^{(u)} - W^{(v)} - (1 - aa')(\beta_u' - \gamma_v)]dudv - (h' + a'W^{(v)})dv^2 = 0. \tag{88}$$

Comparing this with equation (85), we observe that *the axis curves coincide with the ray curves if and only if*

$$W^{(u)} - (\beta_u' - \gamma_v) = 0, \qquad W^{(v)} + (\beta_u' - \gamma_v) = 0. \tag{89}$$

We may use the results just obtained in stating the following theorem.

THEOREM. *If for the parametric net on the surface S_y the developables of the axis and ray congruences correspond, and if the axis and ray curves both form conjugate nets, then the congruences of tangents to the two parametric families C_u and C_v are W-congruences.*

Another geometric criterion that the ray curves form a conjugate net may be obtained as follows. From the second of equations (78), we find at once that

$$\begin{aligned}\rho_v - \beta' \rho &= y_{uv} - \beta' y_u - \gamma y_v + (\beta' \gamma - \gamma_v) y \\ &= z + (\beta' \gamma - \gamma_v) y .\end{aligned} \tag{90}$$

Now, the line $\rho\rho_v$ is the tangent at ρ to the curve $u = \text{const.}$ on the surface S_ρ. The left-hand member of equation (90) is a point on this tangent, and the right-hand member is a point on the axis of the point y. Therefore, *the axis yz of a point on the surface S_y is met in the point*

$$\zeta = z + (\beta' \gamma - \gamma_v) y \tag{91}$$

by the line $\rho\rho_v$, which is tangent to the corresponding curve $u = \text{const.}$ on the surface S_ρ.

Similarly, *the tangent $\sigma\sigma_u$ to a curve $v = \text{const.}$ on the surface S_σ intersects the corresponding axis yz in the point*

$$\zeta' = z + (\beta' \gamma - \beta'_u) y . \tag{92}$$

Consequently, *the points ζ and ζ' coincide if and only if $\beta'_u = \gamma_v$, that is, if and only if the ray curves form a conjugate net.*

In Section 5, we found that the osculating plane to the curve C_v at the point y is tangent at the point

$$\eta = z - (F' - \beta' \gamma + \beta'_u) y \tag{71 bis}$$

to the ruled surface $R^{(u)}$ of the axis congruence which corresponds to the curve C_u. The point η coincides with the point ζ' given by (92) if and only if $F' = 0$. But the vanishing of the invariant F', which is given by equation (59), is a necessary and sufficient condition that the curves $v = \text{const.}$ on S_σ be asymptotics. Hence, *the points η coincide with the points ζ' if and only if the curve $v = \text{const.}$ on the surface S_σ are asymptotic.*

Similarly, *the points η' given by equation (72) coincide with the points ζ if and only if the curves $u = \text{const.}$ on the surface S_ρ are asymptotic.*

A SET OF POSTULATES FOR FIELDS*

BY

NORBERT WIENER

Introduction

Huntington† has developed a set of thirteen independent postulates for fields. These postulates make use of the undefined notions of multiplication and addition. They thus represent a state of the theory of postulates for fields analogous to that prevailing in boolean algebras before the work of Sheffer.‡ Sheffer showed that the three operations of disjunction, conjunction, and negation could be derived by iterating the operation $\bar{a} \odot \bar{b}$, and that the logical constants z and u could be obtained in like manner. It is the purpose of this paper to perform an analogous task for ordinary algebra. Our undefined operation, which we shall symbolize by $x \,@\, y$, will turn out to have the formal properties characteristic of $1 - x/y$.

Postulate set for fields

We assume:

I. A class K,

II. A binary K-rule of combination $@$.

III. The following properties of K and $@$:

1. Whatever x may be, there is a K-element y such that $x \,@\, (y \,@\, y)$ is not a K-element.

2. If x and y are K-elements, but $x \,@\, y$ is not a K-element, there is a K-element z such that $y = z \,@\, z$.

3. Whenever x and y are distinct K-elements, either $x \,@\, y$ or $y \,@\, x$ is a K-element.

4. Whenever x, y, u, v, and their indicated combinations are K-elements, and $x \,@\, y = u \,@\, v$, then $x \,@\, u = y \,@\, v$.

Definition. $Z = x \,@\, x$, if x and $x \,@\, x$ belong to K, and $x \,@\, x$ is unique in the system.

* Presented to the Society, December 30, 1919.

† E. V. Huntington, *Note on the definitions of abstract groups and fields by sets of independent postulates*, these Transactions, vol. 6 (1905), pp. 181–193.

‡ H. M. Sheffer, *A set of five independent postulates for boolean algebras, with application to logical constants*, these Transactions, vol. 14 (1913), pp. 481–488. Sheffer's terminology and method of exposition are followed closely here.

5. If x, y, $x \mathbin{@} y$, and Z are K-elements, and $x \mathbin{@} y = Z$, then $x = y$.

6. If x, y, z, Z, and their indicated combinations belong to K,

$$[(x \mathbin{@} y) \mathbin{@} (Z \mathbin{@} y)] \mathbin{@} z = [(x \mathbin{@} z) \mathbin{@} (Z \mathbin{@} z)] \mathbin{@} y.$$

Definition. $U = Z \mathbin{@} x$, if x, Z, and $Z \mathbin{@} x$ belong to K, and $Z \mathbin{@} x$ is unique in the system.

Definition. $x \odot y = \{U \mathbin{@} \{[(\overline{U \mathbin{@} y} \mathbin{@} U) \mathbin{@} x] \mathbin{@} U\}\} \mathbin{@} U$, if all the indicated expressions belong to K. Otherwise, if Z belongs to K, $x \odot y = Z$.

Definition. $x \sim y = x \odot (y \mathbin{@} x)$.

7. If x, y, u, v, and all their indicated combinations belong to K, we have $(x \sim y) \sim (u \sim v) = (x \sim u) \sim (y \sim v)$.

Definition. $x \oplus y$ is defined as u, the unique K-element for which $x = u \sim y$, if such a unique K-element exists, and as Z, if Z is a K-element, and if there is no K-element u for which $x = u \sim y$.

Classification of postulates 1–7

Postulate 1 is a simple existence-postulate. Postulates 2 and 3, while in form hypothetical existence-postulates, function as K-closing postulates in a limited sense, for they narrow the class of cases where x and y are K-elements, while $x \mathbin{@} y$ fails to be a K-element. Postulates 6 and 7 are simple equivalence-postulates, for they assert that two expressions shall always have the same meaning when they are significant. Postulates 4 and 5 are hypothetical equivalence-postulates, with equivalences for hypotheses. Thus our set consists of one simple existence-postulate, two hypothetical existence-K-closing postulates, two simple equivalence-postulates, and two hypothetical equivalence-postulates.

Consistency of postulates 1–7

With the following interpretation of K and @ postulates 1–7 are satisfied: K consists of all rational numbers, and $x \mathbin{@} y = 1 - x/y$.

Independence of postulates 1–7

With each of the interpretations of K and @ given in (1)–(7) below, all postulates except the one correspondingly numbered are satisfied; that postulate is therefore independent of the remaining six.

(1) K contains only one element m; $m \mathbin{@} m = m$.

(2) K contains only two elements, m and n; $m \mathbin{@} m = m$; $n \mathbin{@} m = n$; $m \mathbin{@} n$ and $n \mathbin{@} n$ do not belong to K.

(3) K contains only two elements, m and n; $m \mathbin{@} m = m$; $n \mathbin{@} n = n$; $m \mathbin{@} n$ and $n \mathbin{@} m$ do not belong to K.

(4) K contains only three elements, $l_4 = l_1$, l_2, and l_3; $l_k \mathbin{@} l_{k+1}$ does not belong to K; otherwise $l_j \mathbin{@} l_k = l_{k+1}$.

(5) K contains only two elements, m and n; $m \mathbin{@} m = m \mathbin{@} n = n \mathbin{@} m = n$; $n \mathbin{@} n$ does not belong to K.

(6) K contains only three elements, l, m, and n; $@$ is defined by the following table (for example: $l \mathbin{@} m = n$; $l \mathbin{@} l$ does not belong to K).

@	l	m	n
l		n	m
m		l	n
n		m	l

(7) K consists of all complex numbers;

$$(x + iy) \mathbin{@} (u + iv) = 1 - \frac{x - iy}{u - iv}.$$

Deductions from the postulates

Theorem I. *K contains at least two elements.*

Theorem II. *K contains an element answering to the definition of Z.*

Theorem III. *If x and y belong to K, and $y \neq Z$, $x \mathbin{@} y$ belongs to K.*

Theorem IV. *K contains no elements of the form $x \mathbin{@} Z$.*

Theorem V. *K contains an element distinct from Z, answering to the definition of U.*

Theorem VI. *If x, y, and z belong to K and $y \neq Z$, $z \neq Z$, then*

$$[(x \mathbin{@} y) \mathbin{@} U] \mathbin{@} z = [(x \mathbin{@} z) \mathbin{@} U] \mathbin{@} y.$$

Theorem VII. *If x is a K-element, $(x \mathbin{@} U) \mathbin{@} U = x$.*

Theorem VIII. *If $y \neq Z$ and $x = y \odot z$, then $z = (x \mathbin{@} y) \mathbin{@} U$, and conversely.*

Theorem IX. *If x and y belong to K, $x \odot y$ belongs to K.*

Theorem X. $x \odot y = y \odot x$.

Theorem XI. *If $x \odot y = Z$, $x = Z$ or $y = Z$, and vice versa.*

Theorem XII. *If x and y belong to K, $x \oplus y$ belongs to K.*

Theorem XIII. *If $x \oplus x = x$, $x = Z$ and vice versa.*

Theorem XIV. *If $u \odot x = u \odot y$ and $u \neq Z$, then $x = y$.*

Theorem XV. $(x \odot y) \odot z = x \odot (y \odot z)$.

Theorem XVI. $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

Theorem XVII. $x \oplus y = y \oplus x$.

Theorem XVIII. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

Theorem XIX. *If x is a K-element, there is a K-element y such that $y \oplus x = Z$.*

Theorem XX. *If x and y are K-elements, there is a K-element u such that $u \oplus x = y$.*

PROOFS OF THE PRECEDING THEOREMS

Proof of I. If K contains only one element x, it follows by 2 that $x @ x = x$. Hence $x @ (x @ x) = x$ and is a K-element. As this contradicts 1, K contains at least two elements.

Proof of II. (1) K contains an element of the form $x @ x$ [by 1 and 2].

If x and y are K-elements, $x @ y$ or $y @ x$ is a K-element [by 3].

If $x @ y$ is a K-element, $x @ y = x @ y$. Then, if $x @ x$ and $y @ y$ are K-elements,

(2) $$x @ x = y @ y .$$

Similarly, if $y @ x$, $x @ x$, and $y @ y$ are K-elements,

(3) $$x @ x = y @ y .$$

Hence K contains a unique term Z [by (1), (2), (3) and the definition of Z].

Proof of III. This theorem follows directly from II and 2.

Proof of IV. This theorem follows directly from II and 1.

Proof of V. $[(Z @ y) @ (Z @ y)] @ z$ belongs to K unless $y = Z$ or $z = Z$ [by III]. Therefore, if $y \neq Z, z \neq Z$,

$$[(Z @ y) @ (Z @ y)] @ z = [(Z @ z) @ (Z @ z)] @ y \text{ [by 6 and II]}.$$

(1) $\therefore Z @ z = Z @ y$ [by II].

(2) $Z @ Z$ is not a member of K [by IV].

(3) U is a member of K [by (1), (2) and the definition of U].

If $Z = U$, then if x is a K-element other than Z, $Z @ x = Z$ [by III and the definition of U]. Then $Z = x$, which is impossible [by 5]. But K contains an element other than Z [by I].

(4) $\therefore Z \neq U$.

Statements (3) and (4) constitute the theorem.

Theorems I–V inclusive will be used in the following proofs without explicit citation.

Proof of VI. This theorem follows directly from V and 6.

Proof of VII. $[(x @ U) @ U] @ x = [(x @ x) @ U] @ U$ [by VI], $= (Z @ U) @ U = U @ U = Z$.

$$\therefore \quad (x @ U) @ U = x .$$

Proof of VIII. Let $x = y \odot z$ and $x \neq Z, y \neq Z, z \neq Z$. Then

$$Z @ U \neq z @ z \text{ [by V, 5]}.$$

$$\therefore \quad U @ z \neq Z @ z \text{ [by 4]}.$$

(1) $$\therefore \quad (U @ z) @ U \neq Z \text{ [by 5]}.$$

$$\therefore \quad [(\overline{U @ z} @ U) @ y] @ U \neq Z .$$

(2) $\therefore \ \{U @ \{[(\overline{U @ z} @ U) @ y] @ U\}\} @ U$ belongs to K.

$x = \{U @ \{[(\overline{U @ z} @ U) @ y] @ U\}\} @ U$ [Definition of $\odot$].

$x @ U = \{\{U @ \{[(\overline{U @ z} @ U) @ y] @ U\}\} @ U\} @ U$

$= U @ \{[(\overline{U @ z} @ U) @ y] @ U\}$.

$\therefore \ U @ x = [(U @ z) @ U] @ y$ [by 4, VII].

$\therefore \ U @ z = y @ x$ [by 4, VII].

$\therefore \ x @ y = z @ U$ [by 4].

(3) $\therefore \ (x @ y) @ U = (z @ U) @ U = z$ [by VII].

(4) If $y = Z$ or $z = Z$, then $x = Z$ [by the definition of $\odot$].

If $x = Z$, then $\{U @ \{[(\overline{U @ z} @ U) @ y] @ U\}\} @ U$ is not a K-element [by (1)].

(5) Then $y = Z$ or $z = Z$ [by (2)]. $\therefore$ If $y \neq Z$, $x = z = Z$ [by (4) and (5)].

(6) Then $(x @ y) @ U = Z = z$.

Statements (3) and (6) constitute the theorem, and all the steps taken are reversible.

Proof of IX. This follows immediately from II and the definition of $\odot$.

Proof of X. Let $x = y \odot z$ and $y \neq Z$, $z \neq Z$, then

$Z = [(x @ y) @ U] @ z$ [by VIII],

$= [(x @ z) @ U] @ y$ [by VI].

(1) Consequently $x = z \odot y$ [by VIII, 5].

If $y = Z$ or $z = Z$, $\{U @ \{[(\overline{U @ z} @ U) @ y] @ U\}\} @ U$ is not a K-element, and by the definition of $\odot$, $y \odot z = z \odot y = Z$.

This fact and (1) constitute the proof of X.

Proof of XI. Suppose $x \odot y = Z$. Then $x = Z$ or

$y = (Z @ x) @ U = U @ U = Z$ [by VIII].

The converse of this proposition follows from the definition of $\odot$.

Proof of XII. If $x \oplus y$ does not belong to K, there are two distinct K-elements, u and v, such that

(1) $x = u \sim y = v \sim y$ [by definition of $\oplus$].

Then either $u \sim y = Z$ or $(u \sim y) \sim (v \sim y) = Z$ [by XI, definition of $\sim$].

Then either $u \sim y = Z$ or $u \sim v = Z$ or $y = Z$ or

$(u \sim v) \sim (y \sim y) = Z$ [by 7].

If $m \sim n = Z$, $m = n$, and if $m = n \neq Z$, $m \sim n = Z$ [by definition of $\sim$].

$$\therefore \quad u = y \quad \text{or} \quad u = v \quad \text{or} \quad y = Z.$$

$$v = y \quad \text{or} \quad v = u \quad \text{or} \quad y = Z.$$

(2) $$\therefore \quad u = v \quad \text{or} \quad y = Z.$$

If $y = Z$, $x = u \odot U = v \odot U$ [by (1)].

(3) Then $u = (u \mathbin{@} U) \mathbin{@} U = (v \mathbin{@} U) \mathbin{@} U = v$ [by VII, VIII, X].

Statements (2) and (3) constitute the theorem.

Proof of XIII. If $x \oplus x = x$ and $x \neq Z$, then $x \sim x = x$ [by definitions of $\odot$, $\sim$, and $\oplus$].

(1) Then $x = x \odot (x \mathbin{@} x) = x \odot Z = Z$ [by XI].

(2) $Z \oplus Z = Z$ [by XI and the definitions of $\sim$ and of $\oplus$].

Statements (1) and (2) constitute the theorem.

Proof of XIV. Suppose $z = u \odot x = u \odot y$ and u is a K-element other than Z. Then $x = (z \mathbin{@} u) \mathbin{@} U = y$ [by VIII].

Proof of XV. Suppose $u = (x \odot y) \odot z$, $v = x \odot y$, and $x \neq Z$, $y \neq Z$, $z \neq Z$. Then $y = (v \mathbin{@} x) \mathbin{@} U$ and $v = (u \mathbin{@} z) \mathbin{@} U$ [by VIII, X].

$$\therefore \quad y = \{[(u \mathbin{@} z) \mathbin{@} U] \mathbin{@} x\} \mathbin{@} U$$

$$= \{[(u \mathbin{@} x) \mathbin{@} U] \mathbin{@} z\} \mathbin{@} U \text{ [by VI].}$$

The steps already taken are reversible, and by retracing them after the interchange of x and z, it appears that if $x \neq Z$, $y \neq Z$, $z \neq Z$,

(1) $$(x \odot y) \odot z = (z \odot y) \odot x = x \odot (y \odot z) \text{ [by X].}$$

If x, y, or $z = Z$, by XI,

(2) $$(x \odot y) \odot z = Z = x \odot (y \odot z).$$

Statements (1) and (2) constitute the theorem.

Proof of XVI.

LEMMA. $x \odot (u \sim v) = (x \odot u) \sim (x \odot v)$, if both sides belong to K, unless $x = Z$.

Proof. Suppose $u \neq Z$, $v \neq Z$. Let $x \odot u = m$ and $x \odot v = n$. Then $m \neq Z$, $n \neq Z$ [by XI]. Then $u = (m \mathbin{@} x) \mathbin{@} U$ and $v = (n \mathbin{@} x) \mathbin{@} U$ [by VIII]. Hence $u \mathbin{@} m = [(m \mathbin{@} x) \mathbin{@} U] \mathbin{@} m = [(m \mathbin{@} m) \mathbin{@} U] \mathbin{@} x$ [by VI] $= U \mathbin{@} x = v \mathbin{@} n$ [by VI, as before].

(1) Therefore $v \mathbin{@} u = n \mathbin{@} m$ [by 4].

$$x \odot (u \sim v) = x \odot [u \odot (v \mathbin{@} u)] \text{ [by definition of } \sim],$$

$$= (x \odot u) \odot (v \mathbin{@} u) \text{ [by XV]},$$

$$= m \odot (n \mathbin{@} m) \text{ [by (1)]},$$

(2) $$= m \sim n \text{ [by definition of } \sim].$$

(3) If $u = Z$, $m = Z$, and neither $x \odot (u \sim v)$ nor $m \sim n$ is a K-element, by the definition of $\sim$.

If $u \neq Z$, and $v = Z$, then $n = Z$, so that $x \odot (u \sim v) = m \odot (v \mathbin{@} u)$ [as before], $= m \odot U = m \odot (n \mathbin{@} m)$

(4) $= m \sim n$ [as before].

Statements (2), (3), and (4) constitute the lemma.

Let $y \oplus z = p$. If $p \neq Z$, then $y = p \sim z$ [def. of $\oplus$, (1) of VIII]. Then $x \odot y = (x \odot p) \sim (x \odot z)$ [by (2) and (4)].

(5) $\therefore\ x \odot p = (x \odot y) \oplus (x \odot z)$ [by XII, def. of $\oplus$].

If $p = Z$, there is no K-element q such that $y = q \sim z$ [def. of $\oplus$]. Then if $x \neq Z$ there is no q such that $x \odot y = (x \odot q) \sim (x \odot z)$ [by the Lemma, XI, XIV].

(6) Then $x \odot p = (x \odot y) \oplus (x \odot z)$ [by the def. of $\oplus$].

(7) If $x = Z$, the theorem becomes, by XII, $Z = Z \oplus Z$, which is an immediate consequence of the definition of $\oplus$.

Statements (5), (6), and (7) constitute the theorem.

Proof of XVII. Suppose $x \oplus y = u$, $u \neq Z$. Then $x = u \sim y$ [def. of $\oplus$], $= u \odot (y \mathbin{@} u)$ [def. of $\sim$].

This proposition is equivalent to $(x \mathbin{@} u) \mathbin{@} U = y \mathbin{@} u$, or to [by VII, VIII]; $(y \mathbin{@} u) \mathbin{@} U = x \mathbin{@} u$ [by VI].

Reversing our former reasoning, $y = u \sim x$, and

(1) $u = y \oplus x$ [by def. of $\oplus$, XII].

(2) If $x \oplus y = Z$, $y \oplus x = Z$ [by def. of $\oplus$, XII, (1)].

Statements (1) and (2) constitute the theorem.

Proof of XVIII. Let $x \oplus y = m$, $y \oplus z = n$, $m \oplus z = u$, and $x \oplus n = v$.

Case I. Suppose x, y, z, m, n, u, and v are all distinct from Z. Then $x = m \sim y$, $z = n \sim y$, $x = v \sim n$, $z = u \sim m$ [by def. of $\oplus$, XVII].

$\therefore\ x \sim z = (m \sim y) \sim (n \sim y) = (v \sim n) \sim (u \sim m)$ [def. of $\sim$].

$\therefore\ m \sim n = (v \sim n) \sim (u \sim m)$ [by 7, def. of $\sim$].

$\therefore\ u \sim m = (v \sim n) \sim (m \sim n)$ [by XVII, defs. of $\oplus$ and $\sim$].

$= v \sim m$ [by 7, def. of $\sim$].

$\therefore\ Z = (u \sim m) \sim (v \sim m)$ [by def. of $\sim$]

$= (u \sim v) \sim (m \sim m)$ [by 7, def. of $\sim$]

$= u \sim v$ [by def. of $\sim$]

$= u \mathbin{@} v$ [by def. of $\sim$, XI].

(1) $\therefore\ v = u$ [by 5].

Case II. Let x, y, or $z = Z$.

(2) $\quad a \oplus Z = Z \oplus a = a$ [by XVII and the defs. of $\oplus$ and $\sim$].

(3) Combining this proposition with XVII, the proof is obvious.

Case III. Let x, y, z, n, u, and v be all distinct from Z, and let $m = Z$. Then $y = n \sim z$, $x = v \sim n$, $z = u \sim Z = u$ [by defs. of $\oplus$ and $\sim$; XVII].

(4) Whatever p, $x \neq p \sim y$. Suppose $u \neq v$. Then

$$\begin{aligned} x &= v \sim n \\ &= (v \sim n) \sim (u \sim u) \text{ [by def. of } \sim], \\ &= (v \sim u) \sim (n \sim u) \text{ [by 7]}, \\ &= (v \sim u) \sim y. \end{aligned} \tag{5}$$

(6) Statements (5) and (4) contradict one another. Hence $u = v$.

(7) *Case IV.* Similarly, if n alone is Z, $u = v$.

(8) *Case V.* It follows from the cases already considered by XII and the definitions of $\sim$ and $\oplus$ that if $u = Z$, $v = Z$, and vice versa.

Statements (1), (3), (6), (7), and (8) constitute the theorem.

Proof of XIX. If a, b, and x are distinct from Z and from one another,

$$\begin{aligned} (a \sim x) \sim (b \sim x) &= (a \sim b) \sim (x \sim x) \text{ [by 7]}, \\ &= (a \sim b) \odot U \text{ [def. of } \sim, U], \\ &= \{U \mathbin{@} \{[(\overline{U \mathbin{@} U} \mathbin{@} U) \mathbin{@} (a \sim b)] \mathbin{@} U\}\} \mathbin{@} U \\ &\qquad \text{[def. of } \odot], \\ &= \{U \mathbin{@} \{[U \mathbin{@} (a \sim b)] \mathbin{@} U\}\} \mathbin{@} U \text{ [def. of } U], \\ &= \{(a \sim b) \mathbin{@} \{[U \mathbin{@} U] \mathbin{@} U\}\} \mathbin{@} U \text{ [by 4, VI]}, \\ &= a \sim b \text{ [by VII]}. \end{aligned} \tag{1}$$

(2) Then $(a \sim x) \sim a = (b \sim x) \sim b$ [by 7]. Now, $(a \sim x) \sim a$ can never be expressed in the form $u \sim x$. For, suppose $(a \sim x) \sim a = u \sim x$. Then $(a \sim x) \sim u = a \sim x$ [by 7].

$$\therefore \quad a \sim x = (a \sim x) \odot [u \mathbin{@} (a \sim x)] \text{ [def. of } \sim]. \tag{3}$$

But we have just seen in (1) how it can be shown that

$$(a \sim x) = (a \sim x) \odot U. \tag{4}$$

Hence, $u \mathbin{@} (a \sim x) = U$ [(3), (4), X, XIV], $= Z \mathbin{@} (a \sim x)$ [def. of U].

$$\therefore \quad U = Z \mathbin{@} u = (a \sim x) \mathbin{@} (a \sim x) = Z.$$

Since this contradicts V,

(5) $$(a \sim x) \sim a \neq u \sim x.$$

From (2) and (5) we conclude that, if $x \neq Z$, there is a unique term

$$\bar{x} = (a \sim x) \sim a,$$

such that whatever u, $\bar{x} \neq u \sim x$.

(6) $\therefore\ Z = \bar{x} \oplus x$ [def. of $\oplus$, XII]

(7) $Z = Z \oplus Z$ [def. of $\oplus$, U, $\sim$, argument of (1)].

Statements (6) and (7) constitute the theorem, unless it is impossible for an a to be found distinct from Z and x. In this case K contains only Z and U. Here the same result may be obtained by a simple computation.

Proof of XX. Let x and y be K-elements. Then either $y = Z$ or $y \sim x$ is a K-element v [by IX, def. of $\sim$].

(1) If $y = Z$, there is a K-element u such that $u \oplus x = y$ [by XIX].

(2) If $y \neq Z$, since $v = y \sim x$, $v \oplus x = y$ [by def. of $\oplus$, XII].

Postulates 1–7 and the definition of a field

In the article already cited, Huntington gives the following reduced set of postulates for a field, the sufficiency of which he proves, but not their independence.

1. If a and b belong to K, $a \oplus b$ belongs to K.
2. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
3. $a \oplus b = b \oplus a$.
4. If a and b belong to K, $a \odot b$ belongs to K.
5. $(a \odot b) \odot c = a \odot (b \odot c)$.
6. $a \odot b = b \odot a$.
7. Either $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$, or
$$(b \oplus c) \odot a = (b \odot a) \oplus (c \odot a).$$
8. Given a and b, there is an x such that $a \oplus x = b$.
9. Given a and b, and $a \oplus a \neq a$, there is ay such that $a \odot y = b$.
10. There are at least two elements in the class.

That the postulates of Huntington are deducible from those of this paper will be seen at once from an inspection of the following table of correspondences. The figures beneath the line refer to Huntington's postulates; those above, to the present article.

Corresponds to	XII	XVIII	XVII	IX	XV	X	XVI	XX	VII, X	I
	1	2	3	4	5	6	7	8	9	10

If for any elements a and b of Huntington's class, we write $a \mathbin{@} b$ for $1 - a/b$, Huntington's set implies set 1–7.

A SINGLE UNDEFINED NOTION FOR COMPLEX ALGEBRA

The laws of complex algebra cannot be expressed entirely in terms of $+$ and $\times$. Huntington* has given a set of postulates for complex algebra in which the undefined notions are addition, multiplication, the class of all complex numbers, the class of all real numbers, and the order of magnitude among real numbers. He has suggested† how all these notions may be derived from that of the absolute value of the difference between two complex numbers, though he has given no set of postulates for this operation. Apart from this operation, which is rather unlike those on which we are accustomed to base sets of postulates in that it always results in a number of a very special sort—a real number, no single operation has been developed which may serve as the sole foundation of complex algebra.

There are other such operations, however, and one of them is closely allied to the one taken for the fundamental notion of this set of postulates. This operation has already been employed to demonstrate the independence of postulate 7. Denoting $a - ib$, the conjugate‡ imaginary of $a + ib$, by $C(a + ib)$, our operation, which we shall term $x * y$, is

$$1 - C\left(\frac{x}{y}\right).$$

We define Z, U, and multiplication in terms of $*$ in the same manner in which they have already been defined in terms of $@$. $C(x)$ is defined as

$$U * [U * (U * x)].$$

We say that x is real if $x = C(x)$. We define $x \wedge y$ as $C(x * y)$, and addition is defined in terms of $\wedge$ as previously in terms of $@$. If x and y are real, x is said to be larger than y when there is a K-element z, the product of a real by itself, which when added to y gives x.

The formulation of a set of postulates for $*$ is a very difficult matter. Postulates 1–6 of this paper may stand unchanged, but the properties of addition and of the series of reals are so intricate as not to lend themselves readily to independence proofs. In general, independence proofs that are satisfactory when addition and multiplication are assumed separately break down on more than one postulate when translated in terms of $*$. The system of $*$, though fundamentally the same as that of the familiar complex algebra, has a technique of its own of considerable difficulty.

* E. V. Huntington, *A set of postulates for ordinary complex algebra*, these Transactions, vol. 6 (1905), pp. 209–229.

† Ibid., note on p. 210.

‡ This use of the conjugate was suggested to me by an unpublished paper of Sheffer.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
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A THEOREM ON MODULAR COVARIANTS*

BY

OLIVE C. HAZLETT

INTRODUCTION

1. Statement of the problem. This paper answers a question which was raised over five years ago, but which has not been answered so far as I know. Miss Sanderson's theorem† on the relation between formal and modular invariants for the Galois Field $GF[p^n]$ of order p^n enabled her to construct covariants of a system S of binary forms in x and y from invariants of this system S and an additional linear form. This is closely analogous to the situation in the theory of algebraic invariants. In the latter theory the converse also is known to be true—that is, we can form all covariants in this manner. In the case of modular invariants, however, we do not obtain all covariants in this way, for the universal covariant $L = x^{p^n}y - xy^{p^n}$ can not be obtained as a modular invariant of a linear form, since it vanishes whenever x and y are in the field $GF[p^n]$, as we suppose the coefficients of our forms to be. In the paper referred to above, Miss Sanderson raised the question as to whether all covariants of a system S can be expressed as polynomials in L and the modular invariants of the system S enlarged by a linear form. The present paper answers this question in the affirmative.

2. Relation to the literature; definitions. Formal invariants of a form under a group G of linear transformations modulo p, a prime, were first considered by Hurwitz.‡ A few years later, Dickson§ introduced the notion of modular invariants of a form or system of forms.

Consider a system of forms $f_1(x_1, \cdots, x_m), \cdots, f_k(x_1, \cdots, x_m)$, and the group G of linear transformations where the coefficients of the transformations are marks of the $GF[p^n]$. A function of the coefficients of the forms having the invariantive property under this group is called a *formal invariant* if the coefficients of the forms are independent variables. If, on the other hand,

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† *Formal modular invariants with an application to binary modular covariants*, these Transactions, vol. 14 (1913), pp. 489–500.

‡ *Ueber höhere Kongruenzen*, Archiv der Mathematik und Physik, ser. 3, vol. 5 (1903), pp. 17–27.

§ *Invariants of binary forms under modular transformations*, these Transactions, vol. 8 (1907), pp. 205–232.

the coefficients of the forms are indeterminates in the $GF[p^n]$, such a function having the invariantive property is called a *modular invariant*. Every formal invariant is a modular invariant, but not every modular invariant is formally invariant. For, if a is any particular coefficient, $a^{p^n} \equiv a$ in the field and thus $a^{p^n} - a$ is a modular invariant, but not a formal invariant. In a formal invariant, we can not replace a^{p^n} by a where a is a coefficient of one of the original forms.

Similarly, we distinguish between formal covariants and modular covariants according as the coefficients of the forms of S are independent variables or indeterminates in the field.

Dickson* has developed a simple and elegant theory of modular invariants. In this paper he proved the finiteness of modular invariants and later† the finiteness of modular covariants. But thus far no theory of formal invariants has been brought forth, though several writers (chiefly Dickson and Glenn) have found fundamental sets of covariants for special cases.

There is, however, an intimate relation between the modular invariants of a system S of forms and the formal (modular) invariants. This is given in Miss Sanderson's theorem mentioned in Article 1, namely, to any modular invariant i of a system of forms under any group G of linear transformations with coefficients in the $GF[p^n]$, there corresponds a formal invariant I under G such that $I \equiv i$ for all sets of values in the field of the coefficients of the system of forms.

This theorem enabled her, as in the classical theory of algebraic invariants, to construct covariants of a system S of binary forms in x and y from invariants of the enlarged system S', where S' consists of the forms of S in the variables ξ and η, and an additional linear form whose coefficients are y and $-x$. As Miss Sanderson‡ points out in her thesis, the universal covariant $L = x^{p^n} y - xy^{p^n}$ can not be obtained as a modular invariant of a linear form, since it vanishes whenever x and y are marks of the $GF[p^n]$, as we suppose the coefficients of our forms to be. She then says, "Whether or not all the (modular) covariants of a system of forms can be expressed as functions of this universal covariant and the (modular) invariants of this system and a linear form is a question as yet unanswered."

While reading Miss Sanderson's paper recently, a method of proof of this

* *General theory of modular invariants*, these Transactions, vol. 10 (1909), pp. 123–158.

† *Proof of the finiteness of modular covariants*, these Transactions, vol. 14 (1913), pp. 299–310.

‡ Dickson, *On invariants and the theory of numbers, Madison Colloquium*, pp. 41–53; *Invariants in the theory of numbers*, these Transactions, vol. 1 (1914), pp. 497–503. Glenn, *Formal modular invariant theory of binary quantics*, ibid., vol. 17 (1916), pp. 545–556; *A fundamental system of formal covariants modulo 2 of the binary cubic*, ibid., vol. 19 (1918), pp. 109–118.

theorem occurred to the writer, and then a proof of a generalization to the case of a system of forms in several cogredient binary variables.

The chief interest of this theorem seems to be in the light it throws on the very difficult and, thus far, unsolved problem of formal covariants, since the theory of modular covariants is a stepping stone from the theory of modular invariants to the theory of formal covariants. For a modular invariant is an invariantive function of certain quantities which are all marks of the field; while a modular covariant is an invariantive function of certain sets of quantities all of which are in the field except one pair (i.e., the variables x and y); finally a formal covariant is an invariantive function of certain quantities of which all are independent variables. The present paper shows the relation between the theory of modular invariants and the theory of modular covariants; it shows a way of basing the theory of modular covariants on the theory of modular invariants, and thus may serve to suggest a way of basing the theory of formal covariants on the theory of modular covariants and hence in turn on the theory of modular invariants.*

Fundamental theorem

3. Preliminary lemma. Consider a system S of binary forms in the variables x and y with the coefficients $a_0, a_1, \cdots, b_0, b_1, \cdots, c_0, c_1, \cdots$. There is one and only one modular invariant which is of degree $\leqq p^n - 1$ in each of the coefficients and which assumes the set of values v_i as the a's, b's, $\cdots$ range over all sets of values $a_0^{(i)}, a_1^{(i)}, \cdots, b_0^{(i)}, b_1^{(i)}, \cdots, c_0^{(i)}, c_1^{(i)}, \cdots$ in the field.† Dickson proves this theorem by applying his general theorem on interpolation in a finite field,‡ viz., within the $GF[p^n]$, there exists one and but one polynomial $\phi(x_1, \cdots, x_k)$ which has each exponent $\leqq p^n - 1$ and which takes prescribed values $v_{x_1, \cdots, x_k}$ for every set of elements $x_1, \cdots, x_k$ in the field.

We may also prove the

Lemma. *If I is a modular invariant of the system S of forms and the linear form $\eta x - \xi y$, then we can make I formally invariant as to ξ and η.*

This lemma is a special case of Miss Sanderson's fundamental theorem already referred to, but her theorem does not furnish a simple formula for constructing an invariant C which is congruent to I and which is formally invariant as to ξ and η.

Let $\phi(a, b, c, \cdots; \xi, \eta) = I$ be a modular invariant of weight w. There is no loss of generality in assuming that ϕ is pseudo-homogeneous in ξ and η of degree d. (We shall say that a function f is pseudo-homogeneous of

* In this connection see Dickson, *Madison Colloquium*, pp. 57–58.

† Dickson, these Transactions, vol. 10, p. 125.

‡ Loc. cit., p. 124.

degree d if, when ξ and η are multiplied by ρ, any non-zero mark of the field, the function f is multiplied by ρ^d; that is, the degrees of the different terms of f differ by integral multiples of $p^n - 1$.) For, if ϕ is not pseudo-homogeneous in ξ and η, it is the sum of a finite number of modular invariants which are pseudo-homogeneous in ξ and η.

First, we construct a function C which is homogeneous in the independent variables ξ and η, and such that $C \equiv I$ whenever ξ and η are in the field. Take

$$C = \sum \left[\phi(a, b, c, \cdots; \kappa, \lambda) \frac{\prod'(\nu\xi - \mu\eta)^d}{\prod'(\nu\kappa - \mu\lambda)^d} \right].$$

Here $\sum$ indicates the sum of all terms of the type indicated, as κ, λ range over the pairs $(-1, 0)$, $(0, 1)$, $(\beta, 1)$, $\cdots$, $(\beta^k, 1)$, $\cdots$ where β is a primitive root of the $GF[p^n]$. Call these the pairs of the set σ. Inside the bracket, Π' indicates the product of all terms of the type indicated, as μ, ν range over all pairs of the set σ except the pair κ, λ occurring in the coefficient $\phi(a, b, c, \cdots; \kappa, \lambda)$ in that bracket. Notice that there are p^n distinct factors in each Π'.

Now, when ξ and η are in the field, they are of the form κ, λ or $\rho\kappa, \rho\lambda$ where κ, λ is a pair of the set σ. In the first case, we evidently have $C \equiv I$; and similarly in the second case, since ϕ is pseudo-homogeneous in ξ and η of degree d.

Next, we wish to show that C is formally invariant as to ξ and η under transformations of the group G. It will be sufficient to prove this for the generators of the group.

The transformation

T_α

$$\xi' = \xi + \alpha\eta,$$
$$\eta' = \eta,$$

of determinant unity, carries $(\xi, \eta) = (-1, 0)$ into itself and interchanges the other pairs of the set σ, and hence interchanges the factors $\nu\xi - \mu\eta$. Also $\phi(a', b', c', \cdots; \kappa', \lambda') = \phi(a, b, c, \cdots; \kappa, \lambda)$, where (κ', λ') is in the set σ if (κ, λ) is, and conversely. Hence C is formally invariant as to ξ and η under T_α.

Under the transformation

T

$$\xi' = -\eta,$$
$$\eta' = \xi,$$

of determinant unity, the pairs of the set σ and the factors $\nu\xi - \mu\eta$ are transformed as follows:—the new $(-1, 0)$ is the old $(0, 1)$, the new $(0, 1)$ is the old $(1, 0)$ and the new $(\beta^k, 1)$ is the old $(1, -\beta^k)$; while $\eta' = \xi$, $\xi' = -\eta$ and $\xi' - \beta^k\eta' = -\beta^k[\xi - (-\beta^{k(p^n-2)})\eta]$. Hence any one of

the pairs of the set σ, say (κ', λ'), is equal to $(A\kappa, A\lambda)$, where (κ, λ) is a pair of the set σ. Also, the corresponding product $\Pi'(\nu'\xi' - \mu'\eta')$ is equal to $\Pi'(\nu\xi - \mu\eta)$ multiplied by ± 1 times a power of β. Call this multiplier B. Hence $\Pi'(\nu'\kappa' - \mu'\lambda') = AB\Pi'(\nu\kappa - \mu\lambda)$. But

$$\begin{aligned}\phi(a', b', c', \cdots; \kappa', \lambda') &= \phi(a, b, c, \cdots; A\kappa, A\lambda)\\ &\equiv A^d \phi(a, b, c, \cdots; \kappa, \lambda).\end{aligned}$$

Therefore, under the transformation T,

$$\phi(a', b', c', \cdots; \kappa', \lambda') \frac{\prod'(\nu'\xi' - \mu'\eta')^d}{\prod'(\nu'\kappa' - \mu'\lambda')^d} \equiv \phi(a, b, c, \cdots; \kappa, \lambda) \frac{\prod'(\nu\xi - \mu\eta)^d}{\prod'(\nu\kappa - \mu\lambda)^d}$$

where ξ and η are independent variables.

Similarly, it may be proved that, under the transformation

$$T'_\gamma \qquad \begin{aligned}\xi' &= \gamma\xi,\\ \eta' &= \eta\end{aligned} \qquad (\gamma \neq 0 \text{ in the field}),$$

C is formally invariant as to ξ and η.

Since the transformations T_α, T, and T'_γ generate the group G, we see that C is a modular invariant of the system which is formally invariant as to ξ and η.

4. Fundamental theorem for a pair of binary variables. Let $K(a, b, c, \cdots; x, y)$ be a modular covariant of the system S of binary forms which is homogeneous of degree d in x and y. Let the original coefficients be $a_0, a_1, \cdots, b_0, b_1, \cdots, c_0, c_1, \cdots$. When we subject the variables x, y to the non-singular linear transformation with coefficients in the $GF[p^n]$,

$$(4) \qquad \begin{aligned}x &= \alpha X + \beta Y,\\ y &= \gamma X + \delta Y,\end{aligned} \qquad \Delta = \begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix} \neq 0,$$

let the coefficients of the transformed forms be respectively $A_0, A_1, \cdots, B_0, B_1, \cdots, C_0, C_1, \cdots$. Then if K is of weight w,

$$(5) \qquad K(A, B, C, \cdots; X, Y) = \begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}^w K(a, b, c, \cdots; x, y).$$

If, now, x', y' and X', Y' are two pairs of variables so related that

$$(4') \qquad \begin{aligned}x' &= \alpha X' + \beta Y',\\ y' &= \gamma X' + \delta Y',\end{aligned}$$

then

$$(5') \qquad K(A, B, C, \cdots; X', Y') = \begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}^w K(a, b, c, \cdots; x', y').$$

But if $\eta x - \xi y$ is a linear form with coefficients which are independent variables, and $\mathrm{H}X - \Xi Y$ is its transform, then

$$\xi = \alpha\frac{\Xi}{\Delta} + \beta\frac{\mathrm{H}}{\Delta}, \qquad \eta = \gamma\frac{\Xi}{\Delta} + \delta\frac{\mathrm{H}}{\Delta}; \tag{6}$$

or ξ, η and Ξ/Δ, H/Δ are two such pairs of variables. Thus

$$K\left(A, B, C, \cdots; \frac{\Xi}{\Delta}, \frac{\mathrm{H}}{\Delta}\right) = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}^{w} K(a, b, c, \cdots; \xi, \eta).$$

Since $K(a, b, c, \cdots; x, y)$ is homogeneous in x and y of degree d, this equation is equivalent to

$$K(A, B, C, \cdots; \Xi, \mathrm{H}) = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}^{w+d} K(a, b, c, \cdots; \xi, \eta). \tag{7}$$

That is, to every equation (5) where x and y are the variables of the system S of binary forms there corresponds an equation (7) where ξ and η are independent variables and η, $-\xi$ are the coefficients of the linear form $\eta x - \xi y$. In particular, (7) holds whenever ξ, η and Ξ, H are marks of the field so related that (6) holds. Let $K(a, b, c, \cdots; \xi, \eta)$ become $I(a, b, c, \cdots; \xi, \eta)$ when ξ and η are in the field. Then, by the foregoing, $I(a, b, c, \cdots; \xi, \eta)$ is a modular invariant of the system S' consisting of the forms of S together with the linear form $\eta x - \xi y$, which is of weight $w + d$.

If $I(\xi, \eta)$ is the same as $K(\xi, \eta)$, our theorem is proved for the covariant K. We shall, accordingly, consider the case where $I(\xi, \eta)$ is not identically equal to $K(\xi, \eta)$.

Let I be congruent to $A_0(a, b, c, \cdots)$ for $\xi = 1$, $\eta = 0$. Then A_0 is the leader of $K(x, y)$. Accordingly consider the function

$$K'(\xi, \eta) = A_0(a, b, c, \cdots)\xi^d + A_1(a, b, c, \cdots)\xi^{d-1}\eta + \cdots + A_d(a, b, c, \cdots)\eta^d,$$

where $A_1, A_2, \cdots, A_d$ are polynomials in a's, b's, c's, $\cdots$ to be determined. In order that this shall be an invariant of S' which is formally invariant as to ξ and η, $A_0, A_1, \cdots, A_d$ must satisfy certain relations.* What these relations are does not concern us here. We do know, however, that these relations are consistent for at least one set of coefficients $A_0, \cdots, A_d$. Accordingly, let $A_1, A_2, \cdots, A_d$ be one set of polynomials satisfying these relations. Then K' and K are both invariants of S' of the same degree in ξ and η which are congruent when $\eta = 0$. Hence $K - K'$ is such an invariant of S' which vanishes for $\eta = 0$. Thus either $K - K'$ is identically zero in

*O. E. Glenn, *The formal modular invariant theory of binary quantics*, these Transactions, vol. 17 (1916), p. 547 et seq.

ξ and η, or η is a factor of $K - K'$. In the latter case, since $K - K'$ is an invariant which is homogeneous in ξ and η, $L(\xi, \eta) = \xi^{p^n}\eta - \xi\eta^{p^n}$ must be a factor of it.

That is, any invariant K of S' which is formally invariant as to ξ and η is of the form

$$K(\xi, \eta) = K'(\xi, \eta) + L(\xi, \eta)K_0(\xi, \eta)$$

where K_0 is a homogeneous polynomial in ξ and η with coefficients in the field. Notice, moreover, that for all invariants of the same degree in ξ and η which are congruent to the same modular invariant I we can use the same invariant K'. Since K, K', and L are invariants of S' which are formally invariant as to ξ and η, K_0 will be an invariant of S' which is formally invariant as to ξ and η and which is of degree $< d$ in ξ and η.

Hence every covariant $K(x, y)$ of the system S is of the form $K'(x, y) + L(x, y)K_0(x, y)$ where $K'(x, y)$ is the same function for every covariant of a given order and with a given leader, and where K_0 is a covariant of S of order $< d$.

Now let $\Sigma(\xi, \eta)$ denote a set of invariants $K(\xi, \eta)$ of S' determined in the following manner. Consider any particular modular invariant I of S' (all of whose exponents are $\leqq p^n - 1$) and the totality of all invariants K which are $\equiv I$ and which in addition are formally invariant as to ξ and η. These invariants $K(\xi, \eta)$ will be of degrees, d, $d + p^n - 1$, $d + 2(p^n - 1)$, $\cdots$, $d + q(p^n - 1)$, $\cdots$ in ξ and η where d is a suitable positive integer. For each degree, choose one invariant K to put in the set $\Sigma(\xi, \eta)$. Do this for every $I \not\equiv 0$, and let $\Sigma(\xi, \eta)$ denote the set of such invariants $K(\xi, \eta)$ and the invariant I which is identically zero. We shall use $\Sigma(x, y)$ to denote this set of functions with ξ, η replaced by x, y. The members of $\Sigma(x, y)$ are covariants of S.

Then, proceeding by induction, we see that every modular covariant of S is a polynomial in $L = x^{p^n}y - xy^{p^n}$ whose coefficients are modular covariants of the set $\Sigma(x, y)$. Moreover, in view of the above argument, this expansion of any covariant K as a polynomial in L is unique for any given set Σ, though the covariants of the set Σ are not uniquely determinable. Thus we have proved

THEOREM I. *Let S be a system of binary forms in the variables x and y and let S' be the system consisting of the forms of S (where the variables are now ξ and η) together with the linear form $\eta x - \xi y$. Then every modular covariant of the system S is a polynomial in L with coefficients which are modular invariants of the system S' chosen form the set Σ.*

This theorem has already been verified by Miss Sanderson* for some special cases including the binary quadratic, modulo 3.

* Loc. cit., p. 491, 499.

Notice that in the proof of this theorem no use was made of the assumption as to the character of K except that it was formally invariant as to x and y—that is, no assumption was made as to whether or not K was formally invariant as to any or all sets of coefficients or a pair of variables cogredient with x and y. In short, we have the following

COROLLARY. *If K is the class of all modular concomitants of the system S which are formally invariant as to certain sets of coefficients and variables but not formally invariant as to x and y, then the theorem tells us how to construct all modular concomitants of S which are formally invariant as to x and y in addition to being formally invariant as to those sets of coefficients and variables with respect to which K is formally invariant.*

5. Generalization to m pairs of binary variables. Consider a system S of forms in the two pairs of binary variables x_1, y_1 and x_2, y_2, and let K be a modular (mixed) concomitant of S under a group G of linear transformations with coefficients in the $GF[p^n]$. Let S' be the system consisting of the forms of S (where x_1, y_1 have been replaced by ξ_1, η_1) together with the additional linear form $l_1 = \eta_1 x_1 - \xi_1 y_1$; and let S'' be the system consisting of the forms of S together with the two additional forms l_1 and $l_2 = \eta_2 x_2 - \xi_2 y_2$. By the theorem of § 4 in its extended form as given in the corollary, K is a polynomial in $L_1 = x^{p^n} y_1 - x_1 y^{p^n}$ and those invariantive functions of x_2, y_2 and of the coefficients of S' which have been made formally invariant as to x_1 and y_1. Applying the corollary to the modular covariants K' of S', we see that K is a polynomial in L_1, $L_2 = x^{p^n} y_2 - x_2 y^{p^n}$ and the modular invariants of S'' which have been made formally invariant as to x_1, y_1 and x_2, y_2.

Proceeding by induction, we prove

THEOREM II. *Let S be a system of binary forms in the m pairs of variables $(x_1, y_1), (x_2, y_2), \cdots, (x_m, y_m)$. Let $S^{(m)}$ be the system consisting of the forms of S (where now the variables are the ξ_i, η_i) together with the linear forms $l_i = \eta_i x_i - \xi_i y_i$ $(i = 1, \cdots, m)$. Then every modular mixed concomitant of S is a polynomial in the $L_i = x^{p^n} y_i - x_i y^{p^n}$ $(i = 1, \cdots, m)$ and in the modular invariants of $S^{(m)}$ which have been made formally invariant as to the x_i, y_i $(i = 1, \cdots, m)$.*

MOUNT HOLYOKE COLLEGE,
SOUTH HADLEY, MASS.